Maple-Worksheet for Gayen's (1949) method based on Bartlett (1946) for computing Edgeworth polynomials for Students t in terms of **cumulants** (!!!) kappa[i]

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Students t-statistic defined by the one-Sample t-statistic
with nu=n-1 degrees of freedom.

Note: Student's t-statistc in Bartlett (1935) and Gayen (1949)
defined with norming sequence 1/(n-1) in the variance estimator.

Edgeworth expansion has the form:

F_n (t) = Phi(t) + \sum_{i=1}^m p_i (t) phi (t)

with

Phi: cdf of standard normal

phi: pdf of standard normal

p_i (t) depends on cumulants kappa[3], ... , kappa[i+2]
of the underlying universe

p_i (t) can be recomputed in terms of moments alpha[3], ... , alpha[i+2]
(see end of program)

References:

Finner, H., Dickhaus, T. (2009).
EDGEWORTH EXPANSIONS AND RATES OF
CONVERGENCE FOR NORMALIZED SUMS:
CHUNG'S 1946 METHOD REVISITED.
Preprint.

Bartlett, M. S. (1935). The effect of non-normality on the t distribution. Proc. Camb. Philos. Soc. 31, 223-231.
Chung, K.-L. (1946). The approximate distribution of Student's statistics. Ann. Math. Stat. 17, 447-465.
Gayen, A. K. (1949). The distribution of 'Student's' t in random samples of any size

drawn from non-normal universes. Biometrika 36, 353-369.
Hall, P. (1987). Edgeworth expansion for Student's t statistic under minimal moment conditions. Ann. Prob. 15, 920-931.
Hall, P. (1992). The bootstrap and Edgeworth expansion. Springer Series in Statistics, New York.
Hsu, P. L. (1945). The approximate distributions of the mean and variance of a sample of independent variables. Ann. Math. Stat. 16, 1-29.
Wallace, D. L. (1958). Asymptotic approximations to distributions. Ann. Math. Stat. 29, 635-654.

```
> restart:
  with(combinat):
```

USER INPUT: number of approximation polynomials needed

(Number_polynomials >= 9 may take a long time)

```
> number_polynomials:=8;
```

$$number\_polynomials := 8 \quad \textbf{(1)}$$

```
> k:=number_polynomials+2;
  tay:=floor(k/3)+1;
```

$$k := 10$$
$$tay := 4 \quad \textbf{(2)}$$

Definition of standard normal pdf

```
> phi:=x->exp(-x^2/2)/sqrt(2*Pi);
```

$$\phi := x \rightarrow \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} \quad \textbf{(3)}$$

Definition Definition of standard normal pdf of Students t with df=nu

```
> f_nu:=(t,nu)->GAMMA( (nu+1)/2 )/GAMMA(nu/2)/sqrt(nu*Pi)/(1+
  t^2/nu)^((nu+1)/2);
  assume(n0>1000,n>1000,S1::real,S2::real);
  assume(t::real,t>0);
```

(4)

$$f\_nu := (t, \nu) \to \frac{\Gamma\left(\frac{1}{2}\,\nu + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\,\nu\right)\sqrt{\nu\,\pi}\left(1 + \frac{t^2}{\nu}\right)^{\frac{1}{2}\,\nu + \frac{1}{2}}} \tag{4}$$

Definition of joint density of (S1,SS2), cf Gayen (1947, p- 355, formula (2.7)

```
> W2:=(n0,S1,SS2)->exp(-(SS2)/2) / sqrt(2*Pi*n)*(SS2-S1^2/n)^((n0
  -2)/2)/(2^(n0/2)*GAMMA(n0/2));
```

$$W2 := (n0, S1, SS2) \to \frac{e^{-\frac{1}{2}\,SS2}\left(SS2 - \frac{S1^2}{n}\right)^{\frac{1}{2}\,n0 - 1}}{\sqrt{2\,\pi\,n}\;2^{\frac{1}{2}\,n0}\,\Gamma\left(\frac{1}{2}\,n0\right)} \tag{5}$$

Definition of derivatives: cf Gayen (1947, p. 355)

```
> DW:=(j,n0,S1,SS2)->piecewise(j>0,diff(W2(n0,S1,SS2),S1$j),j=0,W2
  (n0,S1,SS2));
```

$$DW := (j, n0, S1, SS2) \to piecewise\left(0 < j, \frac{\partial^j}{\partial S1^j}\,W2(n0, S1, SS2), j = 0, W2(n0, S1, SS2)\right) \tag{6}$$

Definition of derivatives: cf Gayen (1947, p. 355), formula (2.9)

```
> j1:='j1';
  DD:=(nu1,nu2,n0,S1,SS2)->piecewise(nu2>0,(-1)^nu2/(2^nu2)*sum(
  (-1)^j1*binomial(nu2,j1)*DW(nu1,n0-2*j1,S1,SS2),j1=0..nu2),
  nu2=0,DW(nu1,n0,S1,SS2));
```

$$j1 := j1 \tag{7}$$

$$DD := (v1, v2, n0, S1, SS2) \to piecewise\left(0 < v2, \frac{(-1)^{v2}\left(\sum_{j1=0}^{v2}(-1)^{j1}\,combinat\text{:-binomial}(v2, j1)\,DW(v1, n0 - 2\,j1, S1, SS2)\right)}{2^{v2}}, v2 = 0, DW(v1, n0, S1, SS2)\right)$$

Sequence A059344: See http://www.research.att.com/~njas/sequences/

This sequence gives the factors for D_1^i D_2^j in the last formula on page 224 in Bartlett (1935)

```
> a:=(nn1, kk1) -> nn1!/(kk1! *(nn1-2*kk1)!);
```

$$a := (nn1, kk1) \rightarrow \frac{nn1!}{kk1!\ (nn1 - 2\,kk1)!} \tag{8}$$

Formal Ansatz of Bartlett for the underlying "universe" pdf

```
> p:=unapply(convert(series(exp(x),x=0,tay),polynom),x);
```

$$p := x \rightarrow 1 + x + \frac{1}{2}\,x^2 + \frac{1}{6}\,x^3 \tag{9}$$

```
> xx:=sum(kappa[r]/GAMMA(r+1)*(-D)^r,r=3..k);
```

$$xx := -\frac{1}{6}\,\kappa_3\,D^3 + \frac{1}{24}\,\kappa_4\,D^4 - \frac{1}{120}\,\kappa_5\,D^5 + \frac{1}{720}\,\kappa_6\,D^6 - \frac{1}{5040}\,\kappa_7\,D^7 + \frac{1}{40320}\,\kappa_8\,D^8 - \frac{1}{362880}\,\kappa_9\,D^9 + \frac{1}{3628800}\,\kappa_{10}\,D^{10} \tag{10}$$

```
> f1:=sort(expand(p(xx)-1),D):
> tay1:=(tay-1)*k;
  i1:='i1':
  for i1 from tay1 to (k+1) by -1 do:
     f1:=subs(D^i1=0,f1):
  end:
```

$$tay1 := 30 \tag{11}$$

```
> f1:=collect(expand(f1), D):
  f1 := sort(f1, D, ascending);
```

$$f1 := -\frac{1}{6}\,\kappa_3\,D^3 + \frac{1}{24}\,\kappa_4\,D^4 - \frac{1}{120}\,\kappa_5\,D^5 + \left(\frac{1}{720}\,\kappa_6 + \frac{1}{72}\,\kappa_3^2\right) D^6 + \left(-\frac{1}{5040}\,\kappa_7 - \frac{1}{144}\,\kappa_3\,\kappa_4\right) D^7 + \left(\frac{1}{1152}\,\kappa_4^2 + \frac{1}{720}\,\kappa_3\,\kappa_5 + \frac{1}{40320}\,\kappa_8\right) D^8 + \left(-\frac{1}{362880}\,\kappa_9 - \frac{1}{2880}\,\kappa_4\,\kappa_5 - \frac{1}{4320}\,\kappa_3\,\kappa_6 - \frac{1}{1296}\,\kappa_3^3\right) D^9 + \left(\frac{1}{30240}\,\kappa_3\,\kappa_7 + \frac{1}{3628800}\,\kappa_{10} + \frac{1}{17280}\,\kappa_4\,\kappa_6 + \frac{1}{28800}\,\kappa_5^2 + \frac{1}{1728}\,\kappa_3^2\,\kappa_4\right) D^{10} \tag{12}$$

```
> i1 := 'i1':
  for i1 from k to 1 by -1 do
    f1 :=subs(D^i1=h(A,B,i1),f1);
  end:
> zaehler := 1:
  if (number_polynomials = 1) then
    ziel := binomial(n, 1)*f1:
  else
    aa := [op(f1)];
```

```
  ziel := binomial(n, 1)*aa[1]:
end:

for r_exp from 2 to number_polynomials do
  obj1:=partition(r_exp);
  for obj in op(obj1) do
    zaehler := zaehler + 1;
    print(zaehler, obj);
    expr := expand(product(aa[obj[ell]], ell=1..nops(obj)));
    for runn from 3 to r_exp+2 do
       mm[runn] := degree(expr, h(A, B, runn));
    end;
    ziel := ziel + binomial(n, nops(obj))*multinomial(sum(mm
[zz], zz=3..r_exp+2), seq(mm[rrr], rrr=3..r_exp+2))*expr;
  end;
end:
```

(13)

2, [1, 1]
3, [2]
4, [1, 1, 1]
5, [1, 2]
6, [3]
7, [1, 1, 1, 1]
8, [1, 1, 2]
9, [2, 2]
10, [1, 3]
11, [4]
12, [1, 1, 1, 1, 1]
13, [1, 1, 1, 2]
14, [1, 2, 2]
15, [1, 1, 3]
16, [2, 3]
17, [1, 4]
18, [5]
19, [1, 1, 1, 1, 1, 1]
20, [1, 1, 1, 1, 2]
21, [1, 1, 2, 2]
22, [2, 2, 2]
23, [1, 1, 1, 3]
24, [1, 2, 3]
25, [3, 3]
26, [1, 1, 4]
27, [2, 4]
28, [1, 5]
29, [6]
30, [1, 1, 1, 1, 1, 1, 1]
31, [1, 1, 1, 1, 1, 2]

32, [1, 1, 1, 2, 2]
33, [1, 2, 2, 2]
34, [1, 1, 1, 1, 3]
35, [1, 1, 2, 3]
36, [2, 2, 3]
37, [1, 3, 3]
38, [1, 1, 1, 4]
39, [1, 2, 4]
40, [3, 4]
41, [1, 1, 5]
42, [2, 5]
43, [1, 6]
44, [7]
45, [1, 1, 1, 1, 1, 1, 1, 1]
46, [1, 1, 1, 1, 1, 1, 2]
47, [1, 1, 1, 1, 2, 2]
48, [1, 1, 2, 2, 2]
49, [2, 2, 2, 2]
50, [1, 1, 1, 1, 1, 3]
51, [1, 1, 1, 2, 3]
52, [1, 2, 2, 3]
53, [1, 1, 3, 3]
54, [2, 3, 3]
55, [1, 1, 1, 1, 4]
56, [1, 1, 2, 4]
57, [2, 2, 4]
58, [1, 3, 4]
59, [4, 4]
60, [1, 1, 1, 5]
61, [1, 2, 5]
62, [3, 5]
63, [1, 1, 6]
64, [2, 6]
65, [1, 7]
66, [8]

Collect the appropriate terms of f1(A;B)^n:

```
> h:=(A,B,jj)->sum(a(jj,ii)*A^(jj-2*ii)*B^ii,ii=0..floor(jj/2));
```

$$h := (A, B, jj) \rightarrow \sum_{ii=0}^{\mathrm{floor}\left(\frac{1}{2} jj\right)} a(jj, ii)\, A^{jj-2\,ii}\, B^{ii} \qquad \textbf{(14)}$$

```
> f2:=ziel:
> C:=expand(simplify(convert(f2,polynom))):
```

```
> with(Groebner):
```

Compute the largest exponent of expressions Aî B^j for A,B

```
> kappa_vec:=[seq(kappa[i],i=3..k)]:
  C1:=sort(C, kappa_vec, plex):
  IA:=degree(C,A);
  IB:=degree(C,B);

  C_1:=collect(C1,kappa_vec,distributed):
  C_2:=[op(C_1)]:
  i_nops:=nops(C_2);
```

$$IA := 24$$
$$IB := 8 \qquad \textbf{(15)}$$
$$i\_nops := 66$$

Compute the coefficients of all relevant factors \prod \kappa_i^j

```
> i:='i':
  for i from 1 to i_nops do:
    h1:=[LeadingTerm(C1, plex(op(kappa_vec)))];
    h2[i,1]:=h1[1];
    h2[i,2]:=h1[2];
    C1:=expand(C1-h1[1]*h1[2]);
    print(i,h2[i,2]);
  end:
```

$$1,\ \kappa_3^8 \qquad \textbf{(16)}$$
$$2,\ \kappa_3^7$$
$$3,\ \kappa_4\,\kappa_3^6$$
$$4,\ \kappa_3^6$$
$$5,\ \kappa_3^5\,\kappa_4$$
$$6,\ \kappa_5\,\kappa_3^5$$
$$7,\ \kappa_3^5$$
$$8,\ \kappa_4^2\,\kappa_3^4$$
$$9,\ \kappa_3^4\,\kappa_4$$
$$10,\ \kappa_3^4\,\kappa_5$$
$$11,\ \kappa_3^4\,\kappa_6$$

12, κ_3^4

13, $\kappa_3^3 \kappa_4^2$

14, $\kappa_3^3 \kappa_4 \kappa_5$

15, $\kappa_3^3 \kappa_4$

16, $\kappa_3^3 \kappa_5$

17, $\kappa_3^3 \kappa_6$

18, $\kappa_3^3 \kappa_7$

19, κ_3^3

20, $\kappa_3^2 \kappa_4^3$

21, $\kappa_3^2 \kappa_4^2$

22, $\kappa_3^2 \kappa_4 \kappa_5$

23, $\kappa_3^2 \kappa_4 \kappa_6$

24, $\kappa_3^2 \kappa_4$

25, $\kappa_3^2 \kappa_5^2$

26, $\kappa_3^2 \kappa_5$

27, $\kappa_3^2 \kappa_6$

28, $\kappa_3^2 \kappa_7$

29, $\kappa_3^2 \kappa_8$

30, κ_3^2

31, $\kappa_3 \kappa_4^3$

32, $\kappa_3 \kappa_4^2 \kappa_5$

33, $\kappa_3 \kappa_4^2$

34, $\kappa_3 \kappa_4 \kappa_5$

35, $\kappa_3 \kappa_4 \kappa_6$

36, $\kappa_3 \kappa_4 \kappa_7$

37, $\kappa_3 \kappa_4$

38, $\kappa_3 \kappa_5^2$

39, $\kappa_3 \kappa_5 \kappa_6$

40, $\kappa_3 \kappa_5$

41, $\kappa_3 \kappa_6$

42, $\kappa_3\,\kappa_7$

43, $\kappa_3\,\kappa_8$

44, $\kappa_3\,\kappa_9$

45, κ_3

46, κ_4^4

47, κ_4^3

48, $\kappa_4^2\,\kappa_5$

49, $\kappa_4^2\,\kappa_6$

50, κ_4^2

51, $\kappa_4\,\kappa_5^2$

52, $\kappa_4\,\kappa_5$

53, $\kappa_4\,\kappa_6$

54, $\kappa_4\,\kappa_7$

55, $\kappa_4\,\kappa_8$

56, κ_4

57, κ_5^2

58, $\kappa_5\,\kappa_6$

59, $\kappa_5\,\kappa_7$

60, κ_5

61, κ_6^2

62, κ_6

63, κ_7

64, κ_8

65, κ_9

66, κ_{10}

Main part of the computation of the polynomials in Edgeworth expansion

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First part: Reduce to relevant factors with respect to order of asymptotics in n
and replace A^i B^j by the corresponding derivatives DD

```
> print("Steps to go:", i_nops);
  print();
```

```
P:=0:
for jh1 from 1 to i_nops do:
 print(jh1);
 field:=[op(h2[jh1,1])]:
 #print(field,field[1]);
 n_asympt_test:=-number_polynomials/2;
 test1:=1;
 for i2 from 1 to nops(field) do:
    n_asympt_test1:=-degree(field[i2],A)/2-degree(field[i2],B)+
degree(field  [i2],n):
   if(n_asympt_test1<-number_polynomials/2) then field[i2]:=0;
test1:=1; fi;
   if(test1=1) then  n_asympt_test:=max(n_asympt_test1,
n_asympt_test);fi;
 end:
 h2[jh1,1]:=sum(field[i3],i3=1..nops(field));

 CC:=convert(expand(h2[jh1,1]*h2[jh1,2]),polynom):
 IA:=degree(CC,A);
 IB:=degree(CC,B);
 #print(IA,IB);
 i:='i';
 j:='j';
 for i from IA to 1 by -1 do:
    for j from IB to 1 by -1 do:
       C1:=algsubs(A^i*B^j=(-1)^j*Dz1(i,j,n+(i+j)*2-1,S1,SS2),
CC);
       CC:=C1;
    end;
 end;

 i:='i';
 for i from IA to 1 by -1 do:
   C1:=algsubs(A^i=Dz1(i,0,n+(i+0)*2-1,S1,SS2),CC);
   CC:=C1;
 end;
 for i from IB to 1 by -1 do:
   C1:=algsubs(B^i=(-1)^i*Dz1(0,i,n+(i+0)*2-1,S1,SS2),CC);
   CC:=C1;
 end:

 CC:=subs(Dz1=DD,CC):
```

```
 g1a:=expand(simplify(CC)):
 g1:=unapply(g1a,S1,SS2):
 hh:=simplify(expand(g1(S1,S2+S1^2/n))):
 hh1:=unapply(hh,S1,S2):
 hh3:=hh1(t*sqrt(n*S2/(n-1)),S2)*sqrt(n*S2/(n-1)):
 hh3:
 hh3a:=unapply(hh3,S2):
 u:='u':
 tt5:=(simplify(expand(int(hh3a(u),u=0..infinity)))):
 tt5a:=unapply(tt5,n,t):
 tt5b:=int(tt5a(n,z1),z1=-infinity..zzz):
 tt5c:=simplify(tt5b/f_nu(zzz,n-1)):

 n_asympt_run:=n_asympt_test;

 py1_new:=0;
 tt5cc:=simplify(tt5c/n^n_asympt_run);

 i:='i';
 for i from 1 to 1000 while(n_asympt_run > -number_polynomials/2
-1/2) do:
   help1:=limit(tt5cc,n=infinity);
   py1_new:=py1_new + n^n_asympt_run*help1;
   n_asympt_run:=n_asympt_run-1;
   tt5cc:=(simplify(tt5cc-help1))*n;
 end:

 #print(py1_new);
 P1:=expand(simplify(py1_new));
 P:=simplify(P+P1);
end:
```

"Steps to go:", 66 **(17)**

1
2
3
4
5
6
7
8
9
10
11
12

Second part: Compute

Polynomials in terms of cumulants kappa[j]:

```
> P1:=unapply(convert(simplify(P),polynom),n):
  assume(lambda>0):
  P1a:=collect(simplify(P1(1/lambda^2)),lambda):
  B1:=expand(simplify(convert(series(P1a ,lambda,
  number_polynomials+1),polynom))):
  B1a:=unapply(B1,lambda):
  B1b:=B1a(1/sqrt(n)):
  B1:=unapply(collect(B1b,{n,op(kappa_vec)}),zzz):

> i:='i':
  help1:=B1(t)*sqrt(n):
  for i from 1 to number_polynomials do:
    PPP[i]:=limit(help1,n=infinity);
    help1:=expand(help1-PPP[i])*sqrt(n);
    tt1:={seq(kappa[i3],i3=3..i+2)}:
    PP[i]:=collect(PPP[i],tt1);
  end:

  i:='i':
  for i from 1 to number_polynomials do;
    PP_cum[i]:=PP[i];
  end;
```

$$PP\_cum_1 := \left(\frac{1}{6} + \frac{t\sim^2}{3}\right)\kappa_3 \tag{18}$$

$$PP\_cum_2 := \left(-\frac{1}{18}\,t\sim^5 + \frac{1}{6}\,t\sim - \frac{1}{9}\,t\sim^3\right)\kappa_3^2 + \left(-\frac{1}{4}\,t\sim + \frac{1}{12}\,t\sim^3\right)\kappa_4$$

$$PP\_cum_3 := \left(\frac{7}{324}\,t\sim^6 - \frac{35}{216}\,t\sim^4 + \frac{1}{162}\,t\sim^8 - \frac{35}{432} - \frac{175}{432}\,t\sim^2\right)\kappa_3^3 + \left(\left(\frac{5}{48} + \frac{5}{8}\,t\sim^2 + \frac{5}{24}\,t\sim^4 - \frac{1}{36}\,t\sim^6\right)\kappa_4 + \frac{t\sim^2}{6} - \frac{t\sim^4}{6} + \frac{1}{24}\right)\kappa_3 + \left(-\frac{1}{40} - \frac{1}{5}\,t\sim^2 - \frac{1}{20}\,t\sim^4\right)\kappa_5$$

$$PP\_cum_4 := \left(-\frac{1}{1944}\,t\sim^{11} - \frac{35}{72}\,t\sim + \frac{5}{108}\,t\sim^7 - \frac{5}{1944}\,t\sim^9 + \frac{25}{108}\,t\sim^5 - \frac{5}{216}\,t\sim^3\right)\kappa_3^4 + \left(\left(\frac{1}{6}\,t\sim^3 + \frac{1}{216}\,t\sim^9 + \frac{29}{24}\,t\sim - \frac{5}{12}\,t\sim^5 - \frac{1}{18}\,t\sim^7\right)\kappa_4 + \frac{t\sim}{4} + \frac{t\sim^7}{9} + \frac{t\sim^3}{18} - \frac{t\sim^5}{36}\right)\kappa_3^2 + \left(\frac{1}{60}\,t\sim^7 + \frac{2}{15}\,t\sim^5 - \frac{1}{2}\,t\sim - \frac{1}{12}\,t\sim^3\right)\kappa_5\,\kappa_3 + \left(-\frac{11}{96}\,t\sim^3 + \frac{7}{96}\,t\sim^5 - \frac{37}{96}\,t\sim\right.$$

$$-\frac{1}{288}\,t\sim^{7}\Big)\,\kappa_4^2+\Big(\frac{1}{3}\,t\sim^{3}+\frac{1}{4}\,t\sim-\frac{1}{12}\,t\sim^{5}\Big)\,\kappa_4+\Big(\frac{1}{18}\,t\sim^{3}+\frac{1}{6}\,t\sim-\frac{1}{45}\,t\sim^{5}\Big)\,\kappa_6$$

$$PP\_cum_5:=\Big(\frac{7007}{6912}\,t\sim^{4}-\frac{143}{2592}\,t\sim^{8}+\frac{1001}{864}\,t\sim^{2}+\frac{1001}{6912}+\frac{1}{29160}\,t\sim^{14}+\frac{1001}{10368}\,t\sim^{6}$$
$$+\frac{13}{58320}\,t\sim^{12}-\frac{143}{19440}\,t\sim^{10}\Big)\,\kappa_3^5+\Big(\Big(-\frac{385}{144}\,t\sim^{4}+\frac{11}{96}\,t\sim^{8}+\frac{11}{1296}\,t\sim^{10}-\frac{385}{1152}$$
$$-\frac{1}{1944}\,t\sim^{12}-\frac{385}{1728}\,t\sim^{6}-\frac{385}{128}\,t\sim^{2}\Big)\,\kappa_4-\frac{35\,t\sim^{4}}{216}-\frac{35}{1728}-\frac{245\,t\sim^{2}}{1728}-\frac{t\sim^{8}}{162}$$
$$+\frac{91\,t\sim^{6}}{324}-\frac{7\,t\sim^{10}}{324}\Big)\,\kappa_3^3+\Big(-\frac{1}{360}\,t\sim^{10}+\frac{49}{480}\,t\sim^{6}+\frac{7}{64}+\frac{77}{64}\,t\sim^{2}-\frac{1}{30}\,t\sim^{8}$$
$$+\frac{35}{32}\,t\sim^{4}\Big)\,\kappa_5\,\kappa_3^2+\Big(\Big(\frac{1}{864}\,t\sim^{10}+\frac{7}{96}\,t\sim^{6}+\frac{245}{192}\,t\sim^{4}+\frac{175}{128}\,t\sim^{2}-\frac{7}{192}\,t\sim^{8}+\frac{35}{256}\Big)$$
$$\kappa_4^2+\Big(\frac{5}{192}+\frac{5}{72}\,t\sim^{8}-\frac{5}{96}\,t\sim^{4}-\frac{29}{72}\,t\sim^{6}+\frac{5}{24}\,t\sim^{2}\Big)\,\kappa_4+\Big(-\frac{7}{288}+\frac{1}{135}\,t\sim^{8}$$
$$-\frac{7}{216}\,t\sim^{6}-\frac{49}{144}\,t\sim^{4}-\frac{49}{144}\,t\sim^{2}\Big)\,\kappa_6+\frac{3}{64}+\frac{19\,t\sim^{2}}{96}+\frac{t\sim^{6}}{8}-\frac{11\,t\sim^{4}}{48}\Big)\,\kappa_3+\Big(-\frac{7}{192}$$
$$+\frac{1}{240}\,t\sim^{8}-\frac{7}{16}\,t\sim^{4}-\frac{7}{16}\,t\sim^{2}-\frac{7}{360}\,t\sim^{6}\Big)\,\kappa_5\,\kappa_4+\Big(\frac{3}{40}\,t\sim^{6}-\frac{1}{160}-\frac{1}{16}\,t\sim^{2}$$
$$+\frac{19}{80}\,t\sim^{4}\Big)\,\kappa_5+\Big(\frac{1}{336}+\frac{3}{56}\,t\sim^{2}+\frac{11}{168}\,t\sim^{4}+\frac{1}{252}\,t\sim^{6}\Big)\,\kappa_7$$

$$PP\_cum_6:=\Big(\frac{7}{8748}\,t\sim^{13}-\frac{1}{65610}\,t\sim^{15}+\frac{35}{4374}\,t\sim^{11}-\frac{805}{972}\,t\sim^{5}-\frac{665}{17496}\,t\sim^{9}$$
$$-\frac{1}{524880}\,t\sim^{17}-\frac{245}{486}\,t\sim^{7}+\frac{3115}{1296}\,t\sim+\frac{665}{486}\,t\sim^{3}\Big)\,\kappa_3^6+\Big(\Big(-\frac{49}{2592}\,t\sim^{11}-\frac{7}{7776}\,t\sim^{13}$$
$$+\frac{665}{288}\,t\sim^{5}-\frac{4585}{864}\,t\sim^{3}+\frac{1}{23328}\,t\sim^{15}+\frac{1225}{864}\,t\sim^{7}+\frac{595}{7776}\,t\sim^{9}-\frac{2485}{288}\,t\sim\Big)\,\kappa_4$$
$$-\frac{65\,t\sim^{3}}{54}+\frac{5\,t\sim^{13}}{1944}+\frac{t\sim^{11}}{486}-\frac{35\,t\sim}{24}-\frac{305\,t\sim^{9}}{1944}-\frac{5\,t\sim^{7}}{27}+\frac{55\,t\sim^{5}}{216}\Big)\,\kappa_3^4+\Big(-\frac{61}{72}\,t\sim^{5}$$
$$+\frac{23}{9}\,t\sim^{3}-\frac{23}{648}\,t\sim^{9}+\frac{2}{405}\,t\sim^{11}-\frac{5}{9}\,t\sim^{7}+\frac{31}{8}\,t\sim+\frac{1}{3240}\,t\sim^{13}\Big)\,\kappa_5\,\kappa_3^3+\Big(\Big(-\frac{265}{192}\,t\sim^{5}$$
$$-\frac{125}{144}\,t\sim^{7}-\frac{25}{1728}\,t\sim^{9}+\frac{7}{864}\,t\sim^{11}-\frac{1}{5184}\,t\sim^{13}+\frac{475}{96}\,t\sim^{3}+\frac{1445}{192}\,t\sim\Big)\,\kappa_4^2+\Big($$
$$-\frac{1}{54}\,t\sim^{11}+\frac{107}{48}\,t\sim-\frac{17}{24}\,t\sim^{5}+\frac{83}{432}\,t\sim^{9}+\frac{19}{12}\,t\sim^{3}+\frac{17}{36}\,t\sim^{7}\Big)\,\kappa_4+\Big(-\frac{10}{9}\,t\sim^{3}$$
$$+\frac{5}{648}\,t\sim^{9}+\frac{7}{36}\,t\sim^{5}-\frac{1}{810}\,t\sim^{11}+\frac{1}{6}\,t\sim^{7}-\frac{37}{24}\,t\sim\Big)\,\kappa_6-\frac{17\,t\sim^{3}}{18}+\frac{5\,t\sim^{7}}{18}-\frac{3\,t\sim}{2}$$
$$-\frac{t\sim^{9}}{6}-\frac{t\sim^{5}}{2}\Big)\,\kappa_3^2+\Big(\Big(\frac{3}{8}\,t\sim^{7}+\frac{7}{12}\,t\sim^{5}-\frac{45}{16}\,t\sim^{3}+\frac{1}{108}\,t\sim^{9}-4\,t\sim-\frac{1}{720}\,t\sim^{11}\Big)\,\kappa_5\,\kappa_4$$
$$+\Big(-\frac{1}{3}\,t\sim^{3}-\frac{1}{5}\,t\sim^{7}-t\sim+\frac{7}{60}\,t\sim^{5}-\frac{1}{20}\,t\sim^{9}\Big)\,\kappa_5+\Big(\frac{5}{12}\,t\sim-\frac{11}{315}\,t\sim^{7}-\frac{1}{30}\,t\sim^{5}$$
$$-\frac{1}{756}\,t\sim^{9}+\frac{1}{3}\,t\sim^{3}\Big)\,\kappa_7\Big)\,\kappa_3+\Big(-\frac{425}{384}\,t\sim+\frac{1}{10368}\,t\sim^{11}-\frac{55}{10368}\,t\sim^{9}+\frac{35}{576}\,t\sim^{7}$$
$$+\frac{35}{576}\,t\sim^{5}-\frac{835}{1152}\,t\sim^{3}\Big)\,\kappa_4^3+\Big(\frac{1}{8}\,t\sim+\frac{9}{32}\,t\sim^{5}+\frac{1}{96}\,t\sim^{3}+\frac{1}{96}\,t\sim^{9}-\frac{13}{96}\,t\sim^{7}\Big)\,\kappa_4^2$$

$$+\left(\left(\frac{5}{6}\,t\sim-\frac{7}{180}\,t\sim^{7}+\frac{7}{12}\,t\sim^{3}+\frac{1}{540}\,t\sim^{9}-\frac{1}{60}\,t\sim^{5}\right)\kappa_6+\frac{t\sim^{7}}{12}+\frac{t\sim^{3}}{12}-\frac{t\sim}{4}\right.$$
$$\left.-\frac{5\,t\sim^{5}}{12}\right)\kappa_4+\left(-\frac{1}{800}\,t\sim^{9}-\frac{47}{1200}\,t\sim^{5}-\frac{13}{600}\,t\sim^{7}+\frac{51}{160}\,t\sim+\frac{29}{120}\,t\sim^{3}\right)\kappa_5^{2}+\left(\right.$$
$$\left.-\frac{13}{90}\,t\sim^{5}-\frac{1}{2}\,t\sim+\frac{2}{45}\,t\sim^{7}-\frac{4}{9}\,t\sim^{3}\right)\kappa_6+\left(-\frac{7}{96}\,t\sim^{3}+\frac{11}{3360}\,t\sim^{7}-\frac{3}{32}\,t\sim\right.$$
$$\left.-\frac{1}{480}\,t\sim^{5}\right)\kappa_8$$

$$PP\_cum_7:=\left(-\frac{1154725}{165888}\,t\sim^{4}-\frac{230945}{497664}+\frac{4199}{559872}\,t\sim^{12}-\frac{1062347}{497664}\,t\sim^{6}+\frac{46189}{373248}\,t\sim^{10}\right.$$
$$+\frac{19}{22044960}\,t\sim^{18}-\frac{323}{4898880}\,t\sim^{16}+\frac{1}{11022480}\,t\sim^{20}-\frac{1615}{1959552}\,t\sim^{14}-\frac{2540395}{497664}\,t\sim^{2}$$
$$\left.+\frac{46189}{248832}\,t\sim^{8}\right)\kappa_3^{7}+\left(\left(-\frac{46189}{124416}\,t\sim^{10}+\frac{17}{7776}\,t\sim^{14}+\frac{85085}{55296}+\frac{85085}{4608}\,t\sim^{2}\right.\right.$$
$$\left.-\frac{1}{349920}\,t\sim^{18}+\frac{221221}{27648}\,t\sim^{6}+\frac{17}{233280}\,t\sim^{16}+\frac{1446445}{55296}\,t\sim^{4}-\frac{17017}{27648}\,t\sim^{8}\right.$$
$$\left.\left.-\frac{221}{15552}\,t\sim^{12}\right)\kappa_4-\frac{13\,t\sim^{16}}{58320}-\frac{7007\,t\sim^{6}}{20736}+\frac{1001}{27648}-\frac{10439\,t\sim^{8}}{20736}+\frac{5005\,t\sim^{2}}{13824}$$
$$+\frac{143\,t\sim^{10}}{1440}+\frac{1703\,t\sim^{12}}{46656}+\frac{47047\,t\sim^{4}}{27648}-\frac{17\,t\sim^{14}}{58320}\Bigg)\kappa_3^{5}+\left(-\frac{1}{1944}\,t\sim^{14}-\frac{25025}{6912}\,t\sim^{6}\right.$$
$$+\frac{715}{5184}\,t\sim^{10}-\frac{1}{38880}\,t\sim^{16}-\frac{35035}{4608}\,t\sim^{2}-\frac{35035}{3072}\,t\sim^{4}+\frac{13}{1944}\,t\sim^{12}-\frac{5005}{9216}$$
$$\left.+\frac{2431}{13824}\,t\sim^{8}\right)\kappa_5\,\kappa_3^{4}+\left(\left(\frac{65}{62208}\,t\sim^{12}+\frac{33605}{124416}\,t\sim^{10}+\frac{7865}{13824}\,t\sim^{8}-\frac{55055}{6912}\,t\sim^{6}\right.\right.$$
$$\left.\left.-\frac{25025}{18432}-\frac{325325}{18432}\,t\sim^{2}+\frac{1}{46656}\,t\sim^{16}-\frac{35}{31104}\,t\sim^{14}-\frac{725725}{27648}\,t\sim^{4}\right)\kappa_4^{2}+\left(\right.\right.$$
$$\left.-\frac{8855}{2304}\,t\sim^{4}+\frac{8855}{6912}\,t\sim^{6}+\frac{451}{432}\,t\sim^{8}-\frac{385}{4608}+\frac{11}{3888}\,t\sim^{14}-\frac{715}{2592}\,t\sim^{10}-\frac{4235}{4608}\,t\sim^{2}\right.$$
$$\left.-\frac{85}{1944}\,t\sim^{12}\right)\kappa_4+\left(\frac{17017}{6912}\,t\sim^{2}+\frac{1}{7290}\,t\sim^{14}+\frac{1001}{6912}-\frac{1573}{38880}\,t\sim^{10}+\frac{7007}{1728}\,t\sim^{4}\right.$$
$$\left.+\frac{71071}{51840}\,t\sim^{6}-\frac{13}{11664}\,t\sim^{12}-\frac{1001}{25920}\,t\sim^{8}\right)\kappa_6+\frac{4949\,t\sim^{6}}{5184}-\frac{2275\,t\sim^{2}}{13824}-\frac{1295\,t\sim^{4}}{6912}$$
$$-\frac{337\,t\sim^{8}}{1296}-\frac{35}{1536}+\frac{7\,t\sim^{12}}{144}-\frac{245\,t\sim^{10}}{2592}\Bigg)\kappa_3^{3}+\left(\left(\frac{1001}{64}\,t\sim^{4}+\frac{7007}{1440}\,t\sim^{6}+\frac{5005}{512}\,t\sim^{2}\right.\right.$$
$$\left.-\frac{13}{6480}\,t\sim^{12}+\frac{1}{4320}\,t\sim^{14}-\frac{143}{1152}\,t\sim^{10}-\frac{143}{640}\,t\sim^{8}+\frac{1001}{1536}\right)\kappa_5\,\kappa_4+\left(\frac{7}{256}\right.$$
$$\left.-\frac{1421}{1920}\,t\sim^{6}+\frac{91}{256}\,t\sim^{2}+\frac{49}{64}\,t\sim^{4}-\frac{349}{960}\,t\sim^{8}+\frac{113}{1440}\,t\sim^{10}+\frac{1}{80}\,t\sim^{12}\right)\kappa_5+\left(\right.$$
$$\left.\left.-\frac{1177}{2880}\,t\sim^{6}-\frac{77}{128}\,t\sim^{2}+\frac{11}{3360}\,t\sim^{8}-\frac{11}{384}+\frac{121}{15120}\,t\sim^{10}-\frac{55}{48}\,t\sim^{4}+\frac{1}{4536}\,t\sim^{12}\right)\kappa_7\right)$$
$$\kappa_3^{2}+\left(\left(\frac{55055}{9216}\,t\sim^{4}-\frac{1859}{13824}\,t\sim^{8}-\frac{715}{20736}\,t\sim^{10}+\frac{35035}{9216}\,t\sim^{2}+\frac{5005}{18432}+\frac{25025}{13824}\,t\sim^{6}\right.\right.$$
$$\left.\left.+\frac{143}{62208}\,t\sim^{12}-\frac{1}{31104}\,t\sim^{14}\right)\kappa_4^{3}+\left(-\frac{81}{256}\,t\sim^{8}+\frac{35}{1024}+\frac{105}{256}\,t\sim^{2}-\frac{1}{192}\,t\sim^{12}\right.\right.$$

$$+\frac{105}{64}\,t\sim^{4}+\frac{71}{576}\,t\sim^{10}-\frac{77}{96}\,t\sim^{6}\Bigg)\kappa_4^{2}+\Bigg(\Bigg(-\frac{77}{768}-\frac{231}{128}\,t\sim^{2}-\frac{77}{24}\,t\sim^{4}+\frac{11}{320}\,t\sim^{8}$$
$$+\frac{77}{4320}\,t\sim^{10}-\frac{77}{72}\,t\sim^{6}-\frac{1}{1620}\,t\sim^{12}\Bigg)\kappa_6+\frac{185\,t\sim^{2}}{768}+\frac{15}{512}+\frac{395\,t\sim^{8}}{576}-\frac{65\,t\sim^{4}}{768}$$
$$-\frac{35\,t\sim^{10}}{288}-\frac{1501\,t\sim^{6}}{1152}\Bigg)\kappa_4+\Bigg(\frac{1}{2400}\,t\sim^{12}-\frac{77}{1280}-\frac{1309}{1280}\,t\sim^{2}-\frac{539}{960}\,t\sim^{6}-\frac{231}{128}\,t\sim^{4}$$
$$+\frac{11}{960}\,t\sim^{8}+\frac{143}{14400}\,t\sim^{10}\Bigg)\kappa_5^{2}+\Bigg(\frac{251}{2160}\,t\sim^{8}+\frac{77}{576}\,t\sim^{4}-\frac{7}{270}\,t\sim^{10}-\frac{7}{1152}$$
$$+\frac{175}{432}\,t\sim^{6}-\frac{7}{72}\,t\sim^{2}\Bigg)\kappa_6+\Bigg(\frac{1}{960}\,t\sim^{8}+\frac{43}{480}\,t\sim^{6}+\frac{15}{64}\,t\sim^{4}+\frac{13}{128}\,t\sim^{2}-\frac{11}{10080}\,t\sim^{10}$$
$$+\frac{1}{256}\Bigg)\kappa_8-\frac{29\,t\sim^{4}}{96}+\frac{35}{768}+\frac{7\,t\sim^{2}}{32}-\frac{5\,t\sim^{8}}{48}+\frac{7\,t\sim^{6}}{24}\Bigg)\kappa_3+\Bigg(\frac{143}{3840}\,t\sim^{8}-\frac{3157}{1536}\,t\sim^{4}$$
$$-\frac{77}{64}\,t\sim^{2}-\frac{77}{120}\,t\sim^{6}+\frac{11}{2160}\,t\sim^{10}-\frac{1}{5760}\,t\sim^{12}-\frac{77}{1024}\Bigg)\kappa_5\,\kappa_4^{2}+\Bigg(\Bigg(-\frac{7}{480}\,t\sim^{10}$$
$$+\frac{259}{720}\,t\sim^{6}-\frac{7}{768}-\frac{49}{384}\,t\sim^{2}+\frac{143}{2880}\,t\sim^{8}-\frac{21}{64}\,t\sim^{4}\Bigg)\kappa_5+\Bigg(\frac{11}{64}\,t\sim^{2}-\frac{13}{3360}\,t\sim^{8}$$
$$-\frac{1}{3024}\,t\sim^{10}+\frac{17}{48}\,t\sim^{4}+\frac{1}{128}+\frac{31}{240}\,t\sim^{6}\Bigg)\kappa_7\Bigg)\kappa_4+\Bigg(\Bigg(\frac{7}{640}-\frac{1}{900}\,t\sim^{10}+\frac{7}{32}\,t\sim^{2}$$
$$+\frac{7}{16}\,t\sim^{4}+\frac{7}{48}\,t\sim^{6}-\frac{1}{480}\,t\sim^{8}\Bigg)\kappa_6-\frac{3\,t\sim^{8}}{32}-\frac{9}{1280}-\frac{23\,t\sim^{2}}{320}+\frac{193\,t\sim^{4}}{640}-\frac{33\,t\sim^{6}}{160}\Bigg)$$
$$\kappa_5+\Bigg(\frac{1}{1344}-\frac{5}{504}\,t\sim^{8}-\frac{79}{504}\,t\sim^{6}-\frac{59}{672}\,t\sim^{4}+\frac{5}{336}\,t\sim^{2}\Bigg)\kappa_7+\Bigg(-\frac{1}{12960}\,t\sim^{8}$$
$$-\frac{1}{3456}-\frac{1}{108}\,t\sim^{2}-\frac{23}{864}\,t\sim^{4}-\frac{19}{1620}\,t\sim^{6}\Bigg)\kappa_9$$

$$PP\_cum_8:=\Bigg(\frac{136675}{31104}\,t\sim^{7}-\frac{407}{419904}\,t\sim^{15}-\frac{535535}{31104}\,t\sim+\frac{140525}{139968}\,t\sim^{9}-\frac{1510355}{93312}\,t\sim^{3}$$
$$+\frac{11}{2519424}\,t\sim^{19}-\frac{1925}{139968}\,t\sim^{11}+\frac{55}{839808}\,t\sim^{17}-\frac{1}{264539520}\,t\sim^{23}-\frac{2695}{139968}\,t\sim^{13}$$
$$-\frac{11}{264539520}\,t\sim^{21}+\frac{105875}{93312}\,t\sim^{5}\Bigg)\kappa_3^{8}+\Bigg(\Bigg(-\frac{1}{209952}\,t\sim^{19}-\frac{1}{5184}\,t\sim^{17}-\frac{175}{192}\,t\sim^{5}$$
$$-\frac{89425}{23328}\,t\sim^{9}+\frac{606725}{7776}\,t\sim^{3}+\frac{423395}{5184}\,t\sim+\frac{35}{432}\,t\sim^{11}+\frac{1}{6298560}\,t\sim^{21}+\frac{31}{17496}\,t\sim^{15}$$
$$-\frac{11375}{648}\,t\sim^{7}+\frac{1435}{23328}\,t\sim^{13}\Bigg)\kappa_4+\frac{3745\,t\sim}{288}-\frac{595\,t\sim^{7}}{972}+\frac{3815\,t\sim^{5}}{1944}+\frac{32585\,t\sim^{9}}{34992}$$
$$+\frac{29\,t\sim^{17}}{1049760}+\frac{t\sim^{19}}{65610}-\frac{683\,t\sim^{15}}{131220}+\frac{2135\,t\sim^{11}}{8748}+\frac{12355\,t\sim^{3}}{972}-\frac{385\,t\sim^{13}}{17496}\Bigg)\kappa_3^{6}$$
$$+\Bigg(\frac{539}{324}\,t\sim^{9}+\frac{1}{583200}\,t\sim^{19}-\frac{7}{324}\,t\sim^{13}-\frac{49}{36}\,t\sim^{5}-\frac{41}{48600}\,t\sim^{15}+\frac{175}{24}\,t\sim^{7}-\frac{679}{18}\,t\sim$$
$$-\frac{10759}{288}\,t\sim^{3}+\frac{1}{24300}\,t\sim^{17}-\frac{49}{6480}\,t\sim^{11}\Bigg)\kappa_5\,\kappa_3^{5}+\Bigg(\Bigg(-\frac{2905}{576}\,t\sim^{5}+\frac{1}{15552}\,t\sim^{15}$$
$$+\frac{35245}{1728}\,t\sim^{7}-\frac{791}{15552}\,t\sim^{13}-\frac{260995}{2304}\,t\sim^{3}+\frac{42875}{10368}\,t\sim^{9}+\frac{7}{62208}\,t\sim^{17}-\frac{1477}{10368}\,t\sim^{11}$$
$$-\frac{818825}{6912}\,t\sim-\frac{1}{559872}\,t\sim^{19}\Bigg)\kappa_4^{2}+\Bigg(-\frac{3787}{7776}\,t\sim^{11}+\frac{553}{7776}\,t\sim^{13}+\frac{1505}{864}\,t\sim^{7}$$

$$+\frac{145}{23328}\,t\sim^{15}-\frac{4445}{864}\,t\sim^{5}-\frac{7}{23328}\,t\sim^{17}-\frac{2975}{72}\,t\sim-\frac{33145}{864}\,t\sim^{3}-\frac{21035}{7776}\,t\sim^{9}\Bigg)\,\kappa_4$$
$$+\Bigg(\frac{91}{14580}\,t\sim^{13}+\frac{5159}{324}\,t\sim^{3}-\frac{791}{324}\,t\sim^{7}+\frac{3409}{216}\,t\sim-\frac{1}{87480}\,t\sim^{17}+\frac{1}{8748}\,t\sim^{15}$$
$$+\frac{49}{14580}\,t\sim^{11}-\frac{847}{1458}\,t\sim^{9}+\frac{553}{324}\,t\sim^{5}\Bigg)\,\kappa_6+\frac{175\,t\sim}{32}+\frac{5\,t\sim^{7}}{432}-\frac{845\,t\sim^{9}}{2592}+\frac{2437\,t\sim^{11}}{7776}$$
$$-\frac{5\,t\sim^{15}}{648}+\frac{35\,t\sim^{13}}{1944}+\frac{3505\,t\sim^{3}}{864}+\frac{535\,t\sim^{5}}{144}\Bigg)\,\kappa_3^4+\Bigg(\Bigg(-\frac{112}{9}\,t\sim^{7}-\frac{10087}{3888}\,t\sim^{9}$$
$$+\frac{4361}{54}\,t\sim^{3}+\frac{497}{72}\,t\sim^{5}+\frac{2625}{32}\,t\sim+\frac{77}{3240}\,t\sim^{13}+\frac{7}{162}\,t\sim^{11}+\frac{1}{3645}\,t\sim^{15}$$
$$-\frac{1}{38880}\,t\sim^{17}\Bigg)\,\kappa_5\,\kappa_4+\Bigg(\frac{152}{9}\,t\sim^{3}-\frac{107}{6480}\,t\sim^{13}-\frac{1}{540}\,t\sim^{15}-\frac{1}{4}\,t\sim^{7}+\frac{1003}{48}\,t\sim$$
$$+\frac{76}{405}\,t\sim^{11}+\frac{419}{144}\,t\sim^{5}+\frac{1379}{1296}\,t\sim^{9}\Bigg)\,\kappa_5+\Bigg(-\frac{373}{72}\,t\sim+\frac{323}{1944}\,t\sim^{9}-\frac{11}{9720}\,t\sim^{13}$$
$$-\frac{7}{8}\,t\sim^{5}-\frac{1169}{216}\,t\sim^{3}+\frac{1}{1080}\,t\sim^{11}+\frac{145}{216}\,t\sim^{7}-\frac{1}{40824}\,t\sim^{15}\Bigg)\,\kappa_7\Bigg)\,\kappa_3^3+\Bigg(\Bigg(-\frac{5915}{864}\,t\sim^{7}$$
$$+\frac{581}{7776}\,t\sim^{11}-\frac{37835}{31104}\,t\sim^{9}-\frac{11}{23328}\,t\sim^{15}+\frac{133}{15552}\,t\sim^{13}+\frac{127225}{2304}\,t\sim+\frac{1}{186624}\,t\sim^{17}$$
$$+\frac{44135}{864}\,t\sim^{3}+\frac{9065}{1728}\,t\sim^{5}\Bigg)\,\kappa_4^3+\Bigg(\frac{5105}{192}\,t\sim^{3}+\frac{6085}{3456}\,t\sim^{9}+\frac{11395}{384}\,t\sim-\frac{175}{144}\,t\sim^{7}$$
$$+\frac{71}{576}\,t\sim^{11}+\frac{1075}{384}\,t\sim^{5}+\frac{1}{864}\,t\sim^{15}-\frac{139}{3456}\,t\sim^{13}\Bigg)\,\kappa_4^2+\Bigg(\Bigg(-\frac{805}{32}\,t\sim-\frac{49}{12960}\,t\sim^{13}$$
$$+\frac{1505}{2592}\,t\sim^{9}-\frac{7}{432}\,t\sim^{11}-\frac{119}{32}\,t\sim^{5}+\frac{1}{9720}\,t\sim^{15}-\frac{3493}{144}\,t\sim^{3}+\frac{49}{18}\,t\sim^{7}\Bigg)\,\kappa_6+\frac{5\,t\sim^{13}}{108}$$
$$-\frac{287\,t\sim}{48}-\frac{151\,t\sim^{5}}{24}+\frac{17\,t\sim^{7}}{12}+\frac{23\,t\sim^{9}}{144}-\frac{29\,t\sim^{3}}{6}-\frac{49\,t\sim^{11}}{108}\Bigg)\,\kappa_4+\Bigg(-\frac{91}{43200}\,t\sim^{13}$$
$$+\frac{1519}{960}\,t\sim^{7}-\frac{3843}{320}\,t\sim-\frac{1}{14400}\,t\sim^{15}-\frac{1393}{960}\,t\sim^{5}+\frac{581}{1728}\,t\sim^{9}+\frac{49}{43200}\,t\sim^{11}$$
$$-\frac{11669}{960}\,t\sim^{3}\Bigg)\,\kappa_5^2+\Bigg(\frac{5}{36}\,t\sim^{7}-\frac{67}{1620}\,t\sim^{11}-\frac{1}{8}\,t\sim^{5}-\frac{73}{16}\,t\sim-\frac{55}{18}\,t\sim^{3}+\frac{1}{162}\,t\sim^{13}$$
$$-\frac{473}{1296}\,t\sim^{9}\Bigg)\,\kappa_6+\Bigg(-\frac{1}{4320}\,t\sim^{11}+\frac{67}{192}\,t\sim^{5}+\frac{151}{96}\,t\sim^{3}+\frac{11}{60480}\,t\sim^{13}-\frac{443}{12096}\,t\sim^{9}$$
$$-\frac{15}{112}\,t\sim^{7}+\frac{99}{64}\,t\sim\Bigg)\,\kappa_8+\frac{21\,t\sim}{4}+\frac{4\,t\sim^{7}}{3}-\frac{23\,t\sim^{9}}{36}+\frac{2\,t\sim^{11}}{9}+\frac{23\,t\sim^{3}}{9}+\frac{7\,t\sim^{5}}{9}\Bigg)\,\kappa_3^2$$
$$+\Bigg(\Bigg(-\frac{427}{17280}\,t\sim^{11}-\frac{6349}{192}\,t\sim+\frac{1043}{1728}\,t\sim^{9}+\frac{1}{17280}\,t\sim^{15}-\frac{7}{3240}\,t\sim^{13}+\frac{4025}{1152}\,t\sim^{7}$$
$$-\frac{11921}{384}\,t\sim^{3}-\frac{413}{96}\,t\sim^{5}\Bigg)\,\kappa_5\,\kappa_4^2+\Bigg(\Bigg(-\frac{101}{2160}\,t\sim^{11}-\frac{163}{216}\,t\sim^{9}-\frac{745}{48}\,t\sim^{3}+\frac{1}{144}\,t\sim^{13}$$
$$+\frac{1}{8}\,t\sim^{7}-\frac{113}{48}\,t\sim^{5}-\frac{41}{2}\,t\sim\Bigg)\,\kappa_5+\Bigg(\frac{13}{7560}\,t\sim^{11}-\frac{263}{3024}\,t\sim^{9}+\frac{79}{16}\,t\sim+\frac{47}{48}\,t\sim^{5}$$
$$+\frac{1}{9072}\,t\sim^{13}-\frac{107}{252}\,t\sim^{7}+\frac{115}{24}\,t\sim^{3}\Bigg)\,\kappa_7\Bigg)\,\kappa_4+\Bigg(\Bigg(-\frac{31}{60}\,t\sim^{7}+\frac{1}{2700}\,t\sim^{13}+\frac{673}{120}\,t\sim^{3}$$
$$+\frac{61}{60}\,t\sim^{5}-\frac{59}{540}\,t\sim^{9}+\frac{23}{4}\,t\sim+\frac{1}{1080}\,t\sim^{11}\Bigg)\,\kappa_6+\frac{3\,t\sim^{9}}{20}+\frac{t\sim^{11}}{10}+\frac{113\,t\sim^{3}}{12}+16\,t\sim$$

$$+\frac{77\,t\sim^5}{20}-\frac{t\sim^7}{20}\Big)\kappa_5+\Big(\frac{19}{630}\,t\sim^7+\frac{1}{2}\,t\sim^3+\frac{1}{189}\,t\sim^{11}-\frac{13}{60}\,t\sim^5+\frac{35}{24}\,t\sim$$
$$+\frac{737}{7560}\,t\sim^9\Big)\kappa_7+\Big(\frac{19}{3402}\,t\sim^9+\frac{1}{38880}\,t\sim^{11}-\frac{23}{72}\,t\sim+\frac{59}{3024}\,t\sim^7-\frac{19}{216}\,t\sim^5$$
$$-\frac{275}{864}\,t\sim^3\Big)\kappa_9\Big)\kappa_3+\Big(-\frac{4165}{6144}\,t\sim^5+\frac{35}{165888}\,t\sim^{13}+\frac{5215}{18432}\,t\sim^7+\frac{6545}{165888}\,t\sim^9$$
$$-\frac{343}{55296}\,t\sim^{11}-\frac{72695}{18432}\,t\sim^3-\frac{1}{497664}\,t\sim^{15}-\frac{30415}{6144}\,t\sim\Big)\kappa_4^4+\Big(\frac{5}{32}\,t\sim^7+\frac{25}{1296}\,t\sim^{11}$$
$$-\frac{775}{384}\,t\sim-\frac{5}{10368}\,t\sim^{13}-\frac{1555}{10368}\,t\sim^9-\frac{625}{288}\,t\sim^3-\frac{115}{1152}\,t\sim^5\Big)\kappa_4^3+\Big(\Big(-\frac{71}{288}\,t\sim^7$$
$$-\frac{1}{12960}\,t\sim^{13}+\frac{37}{8640}\,t\sim^{11}+\frac{829}{192}\,t\sim^3-\frac{73}{1728}\,t\sim^9+\frac{43}{48}\,t\sim^5+\frac{341}{64}\,t\sim\Big)\kappa_6+\frac{19\,t\sim^9}{96}$$
$$+\frac{161\,t\sim}{32}-\frac{t\sim^{11}}{48}+\frac{167\,t\sim^3}{48}-\frac{19\,t\sim^7}{36}+\frac{109\,t\sim^5}{48}\Big)\kappa_4^2+\Big(\Big(\frac{2429}{640}\,t\sim+\frac{393}{640}\,t\sim^5$$
$$+\frac{1}{4800}\,t\sim^{11}+\frac{1151}{320}\,t\sim^3-\frac{163}{480}\,t\sim^7+\frac{1}{9600}\,t\sim^{13}-\frac{301}{5760}\,t\sim^9\Big)\kappa_5^2+\Big(\frac{7}{12}\,t\sim^3$$
$$+\frac{13}{135}\,t\sim^9-\frac{3}{20}\,t\sim^5-\frac{1}{135}\,t\sim^{11}+\frac{5}{12}\,t\sim-\frac{17}{180}\,t\sim^7\Big)\kappa_6+\Big(-\frac{11}{64}\,t\sim^5+\frac{107}{4032}\,t\sim^7$$
$$-\frac{85}{128}\,t\sim^3-\frac{311}{384}\,t\sim+\frac{139}{24192}\,t\sim^9-\frac{11}{40320}\,t\sim^{11}\Big)\kappa_8+\frac{t\sim}{4}-\frac{t\sim^5}{2}+\frac{t\sim^7}{2}-\frac{t\sim^9}{12}$$
$$+\frac{t\sim^3}{3}\Big)\kappa_4+\Big(\frac{49}{16}\,t\sim+\frac{49}{1200}\,t\sim^9+\frac{1}{200}\,t\sim^{11}+\frac{239}{120}\,t\sim^3+\frac{1}{15}\,t\sim^7+\frac{32}{75}\,t\sim^5\Big)\kappa_5^2+\Big($$
$$-\frac{2}{3}\,t\sim-\frac{3}{20}\,t\sim^5-\frac{31}{48}\,t\sim^3+\frac{121}{2520}\,t\sim^7+\frac{31}{3780}\,t\sim^9+\frac{1}{5040}\,t\sim^{11}\Big)\kappa_7\,\kappa_5+\Big(-\frac{11}{24}\,t\sim^3$$
$$+\frac{7}{360}\,t\sim^7-\frac{1}{4050}\,t\sim^{11}-\frac{67}{120}\,t\sim-\frac{13}{120}\,t\sim^5+\frac{7}{1620}\,t\sim^9\Big)\kappa_6^2+\Big(-\frac{1}{15}\,t\sim^9+\frac{13}{18}\,t\sim^3$$
$$+\frac{17}{30}\,t\sim^5+\frac{5}{18}\,t\sim^7+\frac{3}{2}\,t\sim\Big)\kappa_6+\Big(\frac{11}{16}\,t\sim+\frac{43}{3360}\,t\sim^7+\frac{47}{96}\,t\sim^3-\frac{11}{1120}\,t\sim^9$$
$$+\frac{121}{480}\,t\sim^5\Big)\kappa_8+\Big(\frac{7}{120}\,t\sim-\frac{13}{12600}\,t\sim^7+\frac{17}{360}\,t\sim^3+\frac{3}{200}\,t\sim^5-\frac{19}{56700}\,t\sim^9\Big)\kappa_{10}$$

Polynomials in terms of moments alpha[j]:

First step: Computation of Moments from cumulants

```
> j:='j':
  f1_moment:=(t,n)->log(1+sum(kappa[j]/j!*(I*t)^j,j=1..k)):
  f1_moment(t,n):
  equation2:=unapply(convert(series(f1_moment(t,n),t=0,k+1),
  polynom,t),t):
  kappa[1]:=0:
  kappa[2]:=1:
```

```
CoefficientVector(equation2(t),t):
i:='i':
for i from 1 to k do;
  moment[i]:=expand(coeff(equation2(t),t^i)*i!/I^i);
end;
i:='i':
moment_vec:=[seq(moment[i],i=3..k)]:
```

$$moment_1 := 0 \tag{19}$$

$$moment_2 := 1$$

$$moment_3 := \kappa_3$$

$$moment_4 := -3 + \kappa_4$$

$$moment_5 := -10\,\kappa_3 + \kappa_5$$

$$moment_6 := -15\,\kappa_4 + \kappa_6 + 30 - 10\,\kappa_3^2$$

$$moment_7 := \kappa_7 - 21\,\kappa_5 - 35\,\kappa_3\,\kappa_4 + 210\,\kappa_3$$

$$moment_8 := 420\,\kappa_4 - 630 - 28\,\kappa_6 + 560\,\kappa_3^2 - 56\,\kappa_3\,\kappa_5 - 35\,\kappa_4^2 + \kappa_8$$

$$moment_9 := 2520\,\kappa_3\,\kappa_4 - 7560\,\kappa_3 - 84\,\kappa_3\,\kappa_6 + 560\,\kappa_3^3 - 126\,\kappa_4\,\kappa_5 - 36\,\kappa_7 + \kappa_9 + 756\,\kappa_5$$

$$moment_{10} := 3150\,\kappa_4^2 - 18900\,\kappa_4 - 210\,\kappa_4\,\kappa_6 + 4200\,\kappa_3^2\,\kappa_4 - 45\,\kappa_8 + \kappa_{10} + 22680 - 120\,\kappa_3\,\kappa_7 + 1260\,\kappa_6 + 5040\,\kappa_3\,\kappa_5 - 126\,\kappa_5^2 - 37800\,\kappa_3^2$$

```
> B2:=unapply(B1(zzz),zzz,op(kappa_vec)):
  Bartlett_complete1:=unapply(expand(B2(t,op(moment_vec))),zzz,op
  (kappa_vec)):
  Bartlett_complete1(zzz,op(kappa_vec)):
  i:='i':
  alpha_vec:=[seq(alpha[i],i=3..k)]:
  Bartlett_complete:=collect(Bartlett_complete1(t,op(alpha_vec)),
  n):
  i:='i':
  help3:=Bartlett_complete*sqrt(n):
  for i from 1 to number_polynomials do:
    PPP1[i]:=limit(help3,n=infinity);
    help3:=expand(help3-PPP1[i])*sqrt(n);
    tt2:={seq(alpha[i3],i3=3..i+2)}:
    PP[i]:=collect(PPP1[i],tt2);
  end:

  i:='i':
  for i from 1 to number_polynomials do;
    PP_mom[i]:=PP[i];
  end;
```

$$PP\_mom_1 := \alpha_3 \left(\frac{1}{6} + \frac{t\sim^2}{3} \right) \tag{20}$$

$$PP\_mom_2 := \left(-\frac{1}{18}\, t\sim^5 + \frac{1}{6}\, t\sim - \frac{1}{9}\, t\sim^3 \right) \alpha_3^2 + \left(-\frac{1}{4}\, t\sim + \frac{1}{12}\, t\sim^3 \right) \alpha_4 - \frac{t\sim^3}{4} + \frac{3\, t\sim}{4}$$

$$PP\_mom_3 := \left(\frac{7}{324}\, t\sim^6 - \frac{35}{216}\, t\sim^4 + \frac{1}{162}\, t\sim^8 - \frac{35}{432} - \frac{175}{432}\, t\sim^2 \right) \alpha_3^3 + \left(\left(\frac{5}{48} + \frac{5}{8}\, t\sim^2 + \frac{5}{24}\, t\sim^4 - \frac{1}{36}\, t\sim^6 \right) \alpha_4 - \frac{7\, t\sim^4}{24} - \frac{1}{48} + \frac{t\sim^6}{12} + \frac{7\, t\sim^2}{24} \right) \alpha_3 + \left(-\frac{1}{40} - \frac{1}{5}\, t\sim^2 - \frac{1}{20}\, t\sim^4 \right) \alpha_5$$

$$PP\_mom_4 := \left(-\frac{1}{1944}\, t\sim^{11} - \frac{35}{72}\, t\sim + \frac{5}{108}\, t\sim^7 - \frac{5}{1944}\, t\sim^9 + \frac{25}{108}\, t\sim^5 - \frac{5}{216}\, t\sim^3 \right) \alpha_3^4 + \left(\left(\frac{1}{6}\, t\sim^3 + \frac{1}{216}\, t\sim^9 + \frac{29}{24}\, t\sim - \frac{5}{12}\, t\sim^5 - \frac{1}{18}\, t\sim^7 \right) \alpha_4 - \frac{t\sim^3}{6} + \frac{t\sim^7}{9} + \frac{t\sim^5}{9} - \frac{t\sim}{24} - \frac{t\sim^9}{72} \right) \alpha_3^2 + \left(\frac{1}{60}\, t\sim^7 + \frac{2}{15}\, t\sim^5 - \frac{1}{2}\, t\sim - \frac{1}{12}\, t\sim^3 \right) \alpha_5 \alpha_3 + \left(-\frac{11}{96}\, t\sim^3 + \frac{7}{96}\, t\sim^5 - \frac{37}{96}\, t\sim - \frac{1}{288}\, t\sim^7 \right) \alpha_4^2 + \left(\frac{3}{16}\, t\sim^3 + \frac{1}{16}\, t\sim + \frac{1}{48}\, t\sim^7 - \frac{3}{16}\, t\sim^5 \right) \alpha_4 + \left(\frac{1}{18}\, t\sim^3 + \frac{1}{6}\, t\sim - \frac{1}{45}\, t\sim^5 \right) \alpha_6 + \frac{25\, t\sim}{32} + \frac{23\, t\sim^5}{96} - \frac{35\, t\sim^3}{96} - \frac{t\sim^7}{32}$$

$$PP\_mom_5 := \left(\frac{7007}{6912}\, t\sim^4 - \frac{143}{2592}\, t\sim^8 + \frac{1001}{864}\, t\sim^2 + \frac{1001}{6912} + \frac{1}{29160}\, t\sim^{14} + \frac{1001}{10368}\, t\sim^6 + \frac{13}{58320}\, t\sim^{12} - \frac{143}{19440}\, t\sim^{10} \right) \alpha_3^5 + \left(\left(-\frac{385}{144}\, t\sim^4 + \frac{11}{96}\, t\sim^8 + \frac{11}{1296}\, t\sim^{10} - \frac{385}{1152} - \frac{1}{1944}\, t\sim^{12} - \frac{385}{1728}\, t\sim^6 - \frac{385}{128}\, t\sim^2 \right) \alpha_4 + \frac{875\, t\sim^2}{3456} + \frac{t\sim^{12}}{648} + \frac{455}{3456} - \frac{25\, t\sim^{10}}{1296} - \frac{235\, t\sim^8}{2592} + \frac{35\, t\sim^4}{108} + \frac{1309\, t\sim^6}{5184} \right) \alpha_3^3 + \left(-\frac{1}{360}\, t\sim^{10} + \frac{49}{480}\, t\sim^6 + \frac{7}{64} + \frac{77}{64}\, t\sim^2 - \frac{1}{30}\, t\sim^8 + \frac{35}{32}\, t\sim^4 \right) \alpha_5 \alpha_3^2 + \left(\left(\frac{1}{864}\, t\sim^{10} + \frac{7}{96}\, t\sim^6 + \frac{245}{192}\, t\sim^4 + \frac{175}{128}\, t\sim^2 - \frac{7}{192}\, t\sim^8 + \frac{35}{256} \right) \alpha_4^2 + \left(-\frac{25}{64}\, t\sim^2 + \frac{13}{96}\, t\sim^8 - \frac{25}{48}\, t\sim^4 - \frac{65}{384} - \frac{1}{144}\, t\sim^{10} - \frac{43}{144}\, t\sim^6 \right) \alpha_4 + \left(-\frac{7}{288} + \frac{1}{135}\, t\sim^8 - \frac{7}{216}\, t\sim^6 - \frac{49}{144}\, t\sim^4 - \frac{49}{144}\, t\sim^2 \right) \alpha_6 + \frac{149\, t\sim^6}{288} + \frac{161\, t\sim^2}{384} - \frac{35\, t\sim^4}{64} - \frac{109\, t\sim^8}{576} + \frac{49}{768} + \frac{t\sim^{10}}{96} \right) \alpha_3 + \left(-\frac{7}{192} + \frac{1}{240}\, t\sim^8 - \frac{7}{16}\, t\sim^4 - \frac{7}{16}\, t\sim^2 - \frac{7}{360}\, t\sim^6 \right) \alpha_5 \alpha_4 + \left(-\frac{1}{80}\, t\sim^8 + \frac{1}{8}\, t\sim^2 + \frac{1}{20}\, t\sim^6 + \frac{13}{320} + \frac{7}{40}\, t\sim^4 \right) \alpha_5 + \left(\frac{1}{336} + \frac{3}{56}\, t\sim^2 + \frac{11}{168}\, t\sim^4 + \frac{1}{252}\, t\sim^6 \right) \alpha_7$$

$$PP\_mom_6 := \left(\frac{7}{8748}\, t\sim^{13} - \frac{1}{65610}\, t\sim^{15} + \frac{35}{4374}\, t\sim^{11} - \frac{805}{972}\, t\sim^5 - \frac{665}{17496}\, t\sim^9 - \frac{1}{524880}\, t\sim^{17} - \frac{245}{486}\, t\sim^7 + \frac{3115}{1296}\, t\sim + \frac{665}{486}\, t\sim^3 \right) \alpha_3^6 + \left(\left(-\frac{49}{2592}\, t\sim^{11} - \frac{7}{7776}\, t\sim^{13} \right.$$

$$+\frac{665}{288}\,t\sim^{5}-\frac{4585}{864}\,t\sim^{3}+\frac{1}{23328}\,t\sim^{15}+\frac{1225}{864}\,t\sim^{7}+\frac{595}{7776}\,t\sim^{9}-\frac{2485}{288}\,t\sim\Bigg)\,\alpha_4$$
$$+\frac{169\,t\sim^{11}}{7776}-\frac{t\sim^{15}}{7776}-\frac{475\,t\sim^{7}}{864}+\frac{17\,t\sim^{13}}{7776}-\frac{125\,t\sim^{5}}{864}+\frac{235\,t\sim^{3}}{864}-\frac{845\,t\sim^{9}}{7776}$$
$$+\frac{35\,t\sim}{32}\Bigg)\,\alpha_3^4+\Bigg(-\frac{61}{72}\,t\sim^{5}+\frac{23}{9}\,t\sim^{3}-\frac{23}{648}\,t\sim^{9}+\frac{2}{405}\,t\sim^{11}-\frac{5}{9}\,t\sim^{7}+\frac{31}{8}\,t\sim$$
$$+\frac{1}{3240}\,t\sim^{13}\Bigg)\,\alpha_5\,\alpha_3^3+\Bigg(\Bigg(-\frac{265}{192}\,t\sim^{5}-\frac{125}{144}\,t\sim^{7}-\frac{25}{1728}\,t\sim^{9}+\frac{7}{864}\,t\sim^{11}-\frac{1}{5184}\,t\sim^{13}$$
$$+\frac{475}{96}\,t\sim^{3}+\frac{1445}{192}\,t\sim\Bigg)\,\alpha_4^2+\Bigg(\frac{1}{864}\,t\sim^{13}+\frac{25}{24}\,t\sim^{7}-\frac{13}{16}\,t\sim^{3}-\frac{5}{144}\,t\sim^{11}+\frac{5}{32}\,t\sim^{5}$$
$$-\frac{87}{32}\,t\sim+\frac{85}{864}\,t\sim^{9}\Bigg)\,\alpha_4+\Bigg(-\frac{10}{9}\,t\sim^{3}+\frac{5}{648}\,t\sim^{9}+\frac{7}{36}\,t\sim^{5}-\frac{1}{810}\,t\sim^{11}+\frac{1}{6}\,t\sim^{7}$$
$$-\frac{37}{24}\,t\sim\Bigg)\,\alpha_6+\frac{11\,t\sim}{64}+\frac{133\,t\sim^{5}}{576}-\frac{25\,t\sim^{3}}{96}-\frac{365\,t\sim^{9}}{1728}+\frac{43\,t\sim^{11}}{864}+\frac{t\sim^{7}}{48}-\frac{t\sim^{13}}{576}\Bigg)\,\alpha_3^2$$
$$+\Bigg(\Bigg(\frac{3}{8}\,t\sim^{7}+\frac{7}{12}\,t\sim^{5}-\frac{45}{16}\,t\sim^{3}+\frac{1}{108}\,t\sim^{9}-4\,t\sim-\frac{1}{720}\,t\sim^{11}\Bigg)\,\alpha_5\,\alpha_4+\Bigg(-\frac{41}{120}\,t\sim^{7}$$
$$+\frac{1}{240}\,t\sim^{11}-\frac{1}{30}\,t\sim^{5}-\frac{1}{40}\,t\sim^{9}+\frac{9}{8}\,t\sim+\frac{17}{48}\,t\sim^{3}\Bigg)\,\alpha_5+\Bigg(\frac{5}{12}\,t\sim-\frac{11}{315}\,t\sim^{7}-\frac{1}{30}\,t\sim^{5}$$
$$-\frac{1}{756}\,t\sim^{9}+\frac{1}{3}\,t\sim^{3}\Bigg)\,\alpha_7\Bigg)\,\alpha_3+\Bigg(-\frac{425}{384}\,t\sim+\frac{1}{10368}\,t\sim^{11}-\frac{55}{10368}\,t\sim^{9}+\frac{35}{576}\,t\sim^{7}$$
$$+\frac{35}{576}\,t\sim^{5}-\frac{835}{1152}\,t\sim^{3}\Bigg)\,\alpha_4^3+\Bigg(-\frac{1}{1152}\,t\sim^{11}+\frac{43}{128}\,t\sim^{3}+\frac{111}{128}\,t\sim-\frac{41}{192}\,t\sim^{7}$$
$$+\frac{11}{192}\,t\sim^{5}+\frac{35}{1152}\,t\sim^{9}\Bigg)\,\alpha_4^2+\Bigg(\Bigg(\frac{5}{6}\,t\sim-\frac{7}{180}\,t\sim^{7}+\frac{7}{12}\,t\sim^{3}+\frac{1}{540}\,t\sim^{9}-\frac{1}{60}\,t\sim^{5}\Bigg)\,\alpha_6$$
$$+\frac{t\sim^{11}}{384}-\frac{27\,t\sim^{5}}{64}+\frac{21\,t\sim^{7}}{64}-\frac{77\,t\sim^{9}}{1152}+\frac{31\,t\sim^{3}}{128}-\frac{33\,t\sim}{128}\Bigg)\,\alpha_4+\Bigg(-\frac{1}{800}\,t\sim^{9}$$
$$-\frac{47}{1200}\,t\sim^{5}-\frac{13}{600}\,t\sim^{7}+\frac{51}{160}\,t\sim+\frac{29}{120}\,t\sim^{3}\Bigg)\,\alpha_5^2+\Bigg(-\frac{11}{72}\,t\sim^{3}-\frac{1}{180}\,t\sim^{9}-\frac{13}{360}\,t\sim^{5}$$
$$-\frac{3}{8}\,t\sim+\frac{5}{72}\,t\sim^{7}\Bigg)\,\alpha_6+\Bigg(-\frac{7}{96}\,t\sim^{3}+\frac{11}{3360}\,t\sim^{7}-\frac{3}{32}\,t\sim-\frac{1}{480}\,t\sim^{5}\Bigg)\,\alpha_8+\frac{105\,t\sim}{128}$$
$$+\frac{119\,t\sim^{5}}{192}-\frac{185\,t\sim^{3}}{384}+\frac{9\,t\sim^{9}}{128}-\frac{t\sim^{11}}{384}-\frac{65\,t\sim^{7}}{192}$$

$$PP\_mom_7:=\Bigg(-\frac{1154725}{165888}\,t\sim^{4}-\frac{230945}{497664}+\frac{4199}{559872}\,t\sim^{12}-\frac{1062347}{497664}\,t\sim^{6}+\frac{46189}{373248}\,t\sim^{10}$$
$$+\frac{19}{22044960}\,t\sim^{18}-\frac{323}{4898880}\,t\sim^{16}+\frac{1}{11022480}\,t\sim^{20}-\frac{1615}{1959552}\,t\sim^{14}-\frac{2540395}{497664}\,t\sim^{2}$$
$$+\frac{46189}{248832}\,t\sim^{8}\Bigg)\,\alpha_3^7+\Bigg(\Bigg(-\frac{46189}{124416}\,t\sim^{10}+\frac{17}{7776}\,t\sim^{14}+\frac{85085}{55296}+\frac{85085}{4608}\,t\sim^{2}$$
$$-\frac{1}{349920}\,t\sim^{18}+\frac{221221}{27648}\,t\sim^{6}+\frac{17}{233280}\,t\sim^{16}+\frac{1446445}{55296}\,t\sim^{4}-\frac{17017}{27648}\,t\sim^{8}$$
$$-\frac{221}{15552}\,t\sim^{12}\Bigg)\,\alpha_4-\frac{25025\,t\sim^{2}}{6912}-\frac{2431\,t\sim^{8}}{82944}-\frac{181181\,t\sim^{4}}{55296}-\frac{359\,t\sim^{14}}{116640}-\frac{11011}{18432}$$
$$+\frac{91\,t\sim^{12}}{3888}+\frac{148291\,t\sim^{10}}{622080}-\frac{43\,t\sim^{16}}{233280}+\frac{t\sim^{18}}{116640}-\frac{17017\,t\sim^{6}}{9216}\Bigg)\,\alpha_3^5+\Bigg(-\frac{1}{1944}\,t\sim^{14}$$

$$
- \frac{25025}{6912}\, t\sim^{6} + \frac{715}{5184}\, t\sim^{10} - \frac{1}{38880}\, t\sim^{16} - \frac{35035}{4608}\, t\sim^{2} - \frac{35035}{3072}\, t\sim^{4} + \frac{13}{1944}\, t\sim^{12}
- \frac{5005}{9216} + \frac{2431}{13824}\, t\sim^{8}\Big)\, \alpha_5\, \alpha_3^4 + \Big(\Big(\frac{65}{62208}\, t\sim^{12} + \frac{33605}{124416}\, t\sim^{10} + \frac{7865}{13824}\, t\sim^{8}
- \frac{55055}{6912}\, t\sim^{6} - \frac{25025}{18432} - \frac{325325}{18432}\, t\sim^{2} + \frac{1}{46656}\, t\sim^{16} - \frac{35}{31104}\, t\sim^{14} - \frac{725725}{27648}\, t\sim^{4}\Big)\, \alpha_4^2
+ \Big(-\frac{11}{768}\, t\sim^{8} + \frac{1}{192}\, t\sim^{14} - \frac{10505}{20736}\, t\sim^{10} + \frac{11165}{2304}\, t\sim^{6} - \frac{1}{7776}\, t\sim^{16} + \frac{9625}{1024}\, t\sim^{2}
+ \frac{4235}{3072} - \frac{17}{1152}\, t\sim^{12} + \frac{39655}{4608}\, t\sim^{4}\Big)\, \alpha_4 + \Big(\frac{17017}{6912}\, t\sim^{2} + \frac{1}{7290}\, t\sim^{14} + \frac{1001}{6912}
- \frac{1573}{38880}\, t\sim^{10} + \frac{7007}{1728}\, t\sim^{4} + \frac{71071}{51840}\, t\sim^{6} - \frac{13}{11664}\, t\sim^{12} - \frac{1001}{25920}\, t\sim^{8}\Big)\, \alpha_6 - \frac{6407\, t\sim^{8}}{13824}
- \frac{49175\, t\sim^{2}}{55296} - \frac{12635}{55296} + \frac{t\sim^{16}}{5184} + \frac{7751\, t\sim^{10}}{41472} + \frac{2489\, t\sim^{12}}{62208} - \frac{235\, t\sim^{14}}{31104} - \frac{7525\, t\sim^{4}}{27648}
+ \frac{35\, t\sim^{6}}{2592}\Big)\, \alpha_3^3 + \Big(\Big(\frac{1001}{64}\, t\sim^{4} + \frac{7007}{1440}\, t\sim^{6} + \frac{5005}{512}\, t\sim^{2} - \frac{13}{6480}\, t\sim^{12} + \frac{1}{4320}\, t\sim^{14}
- \frac{143}{1152}\, t\sim^{10} - \frac{143}{640}\, t\sim^{8} + \frac{1001}{1536}\Big)\, \alpha_5\, \alpha_4 + \Big(-\frac{1925}{512}\, t\sim^{2} + \frac{1}{180}\, t\sim^{12} - \frac{1281}{640}\, t\sim^{6}
- \frac{1}{1440}\, t\sim^{14} - \frac{7}{2}\, t\sim^{4} + \frac{901}{5760}\, t\sim^{10} - \frac{231}{512} - \frac{11}{384}\, t\sim^{8}\Big)\, \alpha_5 + \Big(-\frac{1177}{2880}\, t\sim^{6} - \frac{77}{128}\, t\sim^{2}
+ \frac{11}{3360}\, t\sim^{8} - \frac{11}{384} + \frac{121}{15120}\, t\sim^{10} - \frac{55}{48}\, t\sim^{4} + \frac{1}{4536}\, t\sim^{12}\Big)\, \alpha_7\Big)\, \alpha_3^2 + \Big(\Big(\frac{55055}{9216}\, t\sim^{4}
- \frac{1859}{13824}\, t\sim^{8} - \frac{715}{20736}\, t\sim^{10} + \frac{35035}{9216}\, t\sim^{2} + \frac{5005}{18432} + \frac{25025}{13824}\, t\sim^{6} + \frac{143}{62208}\, t\sim^{12}
- \frac{1}{31104}\, t\sim^{14}\Big)\, \alpha_4^3 + \Big(\frac{1}{3456}\, t\sim^{14} - \frac{12425}{3072}\, t\sim^{4} - \frac{103}{6912}\, t\sim^{12} + \frac{161}{1536}\, t\sim^{8} + \frac{1141}{6912}\, t\sim^{10}
- \frac{1155}{2048} - \frac{4375}{1024}\, t\sim^{2} - \frac{3521}{1536}\, t\sim^{6}\Big)\, \alpha_4^2 + \Big(\Big(-\frac{77}{768} - \frac{231}{128}\, t\sim^{2} - \frac{77}{24}\, t\sim^{4} + \frac{11}{320}\, t\sim^{8}
+ \frac{77}{4320}\, t\sim^{10} - \frac{77}{72}\, t\sim^{6} - \frac{1}{1620}\, t\sim^{12}\Big)\, \alpha_6 - \frac{793\, t\sim^{10}}{2304} + \frac{899\, t\sim^{8}}{1536} + \frac{253\, t\sim^{12}}{6912}
+ \frac{1405\, t\sim^{2}}{1024} + \frac{955\, t\sim^{4}}{3072} + \frac{565\, t\sim^{6}}{4608} - \frac{t\sim^{14}}{1152} + \frac{1805}{6144}\Big)\, \alpha_4 + \Big(\frac{1}{2400}\, t\sim^{12} - \frac{77}{1280}
- \frac{1309}{1280}\, t\sim^{2} - \frac{539}{960}\, t\sim^{6} - \frac{231}{128}\, t\sim^{4} + \frac{11}{960}\, t\sim^{8} + \frac{143}{14400}\, t\sim^{10}\Big)\, \alpha_5^2 + \Big(\frac{97}{8640}\, t\sim^{8}
+ \frac{91}{144}\, t\sim^{6} + \frac{1225}{1152}\, t\sim^{2} + \frac{1}{540}\, t\sim^{12} + \frac{203}{192}\, t\sim^{4} + \frac{77}{768} - \frac{163}{4320}\, t\sim^{10}\Big)\, \alpha_6 + \Big(\frac{1}{960}\, t\sim^{8}
+ \frac{43}{480}\, t\sim^{6} + \frac{15}{64}\, t\sim^{4} + \frac{13}{128}\, t\sim^{2} - \frac{11}{10080}\, t\sim^{10} + \frac{1}{256}\Big)\, \alpha_8 - \frac{2807\, t\sim^{4}}{3072} + \frac{39}{2048}
- \frac{4627\, t\sim^{8}}{4608} + \frac{1405\, t\sim^{2}}{3072} + \frac{t\sim^{14}}{1152} + \frac{821\, t\sim^{10}}{2304} - \frac{29\, t\sim^{12}}{768} + \frac{5987\, t\sim^{6}}{4608}\Big)\, \alpha_3
+ \Big(\frac{143}{3840}\, t\sim^{8} - \frac{3157}{1536}\, t\sim^{4} - \frac{77}{64}\, t\sim^{2} - \frac{77}{120}\, t\sim^{6} + \frac{11}{2160}\, t\sim^{10} - \frac{1}{5760}\, t\sim^{12}
- \frac{77}{1024}\Big)\, \alpha_5\, \alpha_4^2 + \Big(\Big(-\frac{33}{640}\, t\sim^{8} + \frac{1043}{768}\, t\sim^{4} + \frac{1}{960}\, t\sim^{12} + \frac{77}{512} + \frac{63}{80}\, t\sim^{6}
$$

$$
\begin{aligned}
&- \frac{31}{1440}\, t\sim^{10} + \frac{175}{128}\, t\sim^{2}\Big)\, \alpha_5 + \Big(\frac{11}{64}\, t\sim^{2} - \frac{13}{3360}\, t\sim^{8} - \frac{1}{3024}\, t\sim^{10} + \frac{17}{48}\, t\sim^{4} + \frac{1}{128} \\
&+ \frac{31}{240}\, t\sim^{6}\Big)\, \alpha_7\Big)\, \alpha_4 + \Big(\Big(\frac{7}{640} - \frac{1}{900}\, t\sim^{10} + \frac{7}{32}\, t\sim^{2} + \frac{7}{16}\, t\sim^{4} + \frac{7}{48}\, t\sim^{6} \\
&- \frac{1}{480}\, t\sim^{8}\Big)\, \alpha_6 - \frac{49\, t\sim^{8}}{768} + \frac{17\, t\sim^{10}}{480} - \frac{361}{5120} - \frac{281\, t\sim^{2}}{640} - \frac{59\, t\sim^{6}}{480} - \frac{143\, t\sim^{4}}{2560} \\
&- \frac{t\sim^{12}}{640}\Big)\, \alpha_5 + \Big(\frac{1}{224}\, t\sim^{8} - \frac{75}{448}\, t\sim^{2} - \frac{11}{896} - \frac{43}{224}\, t\sim^{4} + \frac{1}{1008}\, t\sim^{10} - \frac{41}{336}\, t\sim^{6}\Big)\, \alpha_7 \\
&+ \Big(-\frac{1}{12960}\, t\sim^{8} - \frac{1}{3456} - \frac{1}{108}\, t\sim^{2} - \frac{23}{864}\, t\sim^{4} - \frac{19}{1620}\, t\sim^{6}\Big)\, \alpha_9
\end{aligned}
$$

$$
\begin{aligned}
PP\_mom_8 := {} & \frac{1659\, t\sim}{2048} + \Big(-\frac{1139}{960}\, t\sim^{3} + \frac{649}{9600}\, t\sim^{9} + \frac{497}{2400}\, t\sim^{7} - \frac{1071}{640}\, t\sim - \frac{1493}{9600}\, t\sim^{5} \\
&+ \frac{1}{4800}\, t\sim^{11} - \frac{1}{3200}\, t\sim^{13}\Big)\, \alpha_5^2 + \Big(\Big(\frac{2429}{640}\, t\sim + \frac{393}{640}\, t\sim^{5} + \frac{1}{4800}\, t\sim^{11} + \frac{1151}{320}\, t\sim^{3} \\
&- \frac{163}{480}\, t\sim^{7} + \frac{1}{9600}\, t\sim^{13} - \frac{301}{5760}\, t\sim^{9}\Big)\, \alpha_5^2 + \Big(\frac{7}{54}\, t\sim^{9} + \frac{1}{2160}\, t\sim^{13} - \frac{49}{80}\, t\sim^{5} + \frac{11}{40}\, t\sim^{7} \\
&- \frac{35}{8}\, t\sim - \frac{139}{48}\, t\sim^{3} - \frac{13}{720}\, t\sim^{11}\Big)\, \alpha_6 + \Big(-\frac{11}{64}\, t\sim^{5} + \frac{107}{4032}\, t\sim^{7} - \frac{85}{128}\, t\sim^{3} - \frac{311}{384}\, t\sim \\
&+ \frac{139}{24192}\, t\sim^{9} - \frac{11}{40320}\, t\sim^{11}\Big)\, \alpha_8 + \frac{27\, t\sim}{512} - \frac{423\, t\sim^{5}}{512} + \frac{181\, t\sim^{3}}{512} - \frac{303\, t\sim^{9}}{512} + \frac{229\, t\sim^{11}}{1536} \\
&+ \frac{1465\, t\sim^{7}}{1536} - \frac{53\, t\sim^{13}}{4608} + \frac{t\sim^{15}}{4608}\Big)\, \alpha_4 + \Big(\frac{136675}{31104}\, t\sim^{7} - \frac{407}{419904}\, t\sim^{15} - \frac{535535}{31104}\, t\sim \\
&+ \frac{140525}{139968}\, t\sim^{9} - \frac{1510355}{93312}\, t\sim^{3} + \frac{11}{2519424}\, t\sim^{19} - \frac{1925}{139968}\, t\sim^{11} + \frac{55}{839808}\, t\sim^{17} \\
&- \frac{1}{264539520}\, t\sim^{23} - \frac{2695}{139968}\, t\sim^{13} - \frac{11}{264539520}\, t\sim^{21} + \frac{105875}{93312}\, t\sim^{5}\Big)\, \alpha_3^8 + \Big(\Big(\\
&- \frac{71}{288}\, t\sim^{7} - \frac{1}{12960}\, t\sim^{13} + \frac{37}{8640}\, t\sim^{11} + \frac{829}{192}\, t\sim^{3} - \frac{73}{1728}\, t\sim^{9} + \frac{43}{48}\, t\sim^{5} + \frac{341}{64}\, t\sim\Big)\, \alpha_6 \\
&+ \frac{4067\, t\sim^{9}}{9216} + \frac{401\, t\sim^{5}}{3072} - \frac{1963\, t\sim^{7}}{9216} + \frac{179\, t\sim^{13}}{27648} - \frac{2183\, t\sim}{1024} - \frac{749\, t\sim^{3}}{1024} - \frac{t\sim^{15}}{9216} \\
&- \frac{317\, t\sim^{11}}{3072}\Big)\, \alpha_4^2 + \Big(-\frac{77}{41472}\, t\sim^{13} - \frac{2165}{4608}\, t\sim^{7} + \frac{1619}{41472}\, t\sim^{11} + \frac{2975}{512}\, t\sim + \frac{2825}{4608}\, t\sim^{5} \\
&+ \frac{16745}{4608}\, t\sim^{3} + \frac{1}{41472}\, t\sim^{15} - \frac{7915}{41472}\, t\sim^{9}\Big)\, \alpha_4^3 + \Big(-\frac{4165}{6144}\, t\sim^{5} + \frac{35}{165888}\, t\sim^{13} \\
&+ \frac{5215}{18432}\, t\sim^{7} + \frac{6545}{165888}\, t\sim^{9} - \frac{343}{55296}\, t\sim^{11} - \frac{72695}{18432}\, t\sim^{3} - \frac{1}{497664}\, t\sim^{15} - \frac{30415}{6144}\, t\sim\Big) \\
&\alpha_4^4 + \Big(-\frac{11}{24}\, t\sim^{3} + \frac{7}{360}\, t\sim^{7} - \frac{1}{4050}\, t\sim^{11} - \frac{67}{120}\, t\sim - \frac{13}{120}\, t\sim^{5} + \frac{7}{1620}\, t\sim^{9}\Big)\, \alpha_6^2 + \Big(\\
&- \frac{1219}{8640}\, t\sim^{9} - \frac{1}{1440}\, t\sim^{13} - \frac{1}{60}\, t\sim^{5} + \frac{199}{8640}\, t\sim^{11} + \frac{197}{576}\, t\sim^{3} + \frac{59}{64}\, t\sim + \frac{23}{288}\, t\sim^{7}\Big)\, \alpha_6 \\
&+ \Big(\frac{11}{13440}\, t\sim^{11} - \frac{23}{1920}\, t\sim^{9} + \frac{137}{384}\, t\sim^{3} + \frac{63}{128}\, t\sim - \frac{137}{6720}\, t\sim^{7} + \frac{89}{960}\, t\sim^{5}\Big)\, \alpha_8 \\
&+ \Big(\frac{7}{120}\, t\sim - \frac{13}{12600}\, t\sim^{7} + \frac{17}{360}\, t\sim^{3} + \frac{3}{200}\, t\sim^{5} - \frac{19}{56700}\, t\sim^{9}\Big)\, \alpha_{10} + \Big(\Big(-\frac{427}{17280}\, t\sim^{11}
\end{aligned}
$$

$$
-\frac{6349}{192}\,t\sim+\frac{1043}{1728}\,t\sim^{9}+\frac{1}{17280}\,t\sim^{15}-\frac{7}{3240}\,t\sim^{13}+\frac{4025}{1152}\,t\sim^{7}-\frac{11921}{384}\,t\sim^{3}
-\frac{413}{96}\,t\sim^{5}\Bigg)\,\alpha_5\,\alpha_4^{2}+\Bigg(\Bigg(-\frac{1}{2880}\,t\sim^{15}+\frac{43}{4320}\,t\sim^{13}+\frac{127}{96}\,t\sim^{5}+\frac{893}{64}\,t\sim^{3}+\frac{151}{2880}\,t\sim^{11}
-\frac{509}{432}\,t\sim^{9}+21\,t\sim-\frac{577}{192}\,t\sim^{7}\Bigg)\,\alpha_5+\Bigg(\frac{13}{7560}\,t\sim^{11}-\frac{263}{3024}\,t\sim^{9}+\frac{79}{16}\,t\sim+\frac{47}{48}\,t\sim^{5}
+\frac{1}{9072}\,t\sim^{13}-\frac{107}{252}\,t\sim^{7}+\frac{115}{24}\,t\sim^{3}\Bigg)\,\alpha_7\Bigg)\,\alpha_4+\Bigg(\Bigg(-\frac{31}{60}\,t\sim^{7}+\frac{1}{2700}\,t\sim^{13}+\frac{673}{120}\,t\sim^{3}
+\frac{61}{60}\,t\sim^{5}-\frac{59}{540}\,t\sim^{9}+\frac{23}{4}\,t\sim+\frac{1}{1080}\,t\sim^{11}\Bigg)\,\alpha_6-\frac{217\,t\sim^{7}}{1920}-\frac{23\,t\sim^{13}}{1440}-\frac{177\,t\sim}{64}
+\frac{t\sim^{15}}{1920}+\frac{1679\,t\sim^{9}}{2880}+\frac{7\,t\sim^{5}}{20}-\frac{257\,t\sim^{3}}{384}+\frac{71\,t\sim^{11}}{5760}\Bigg)\,\alpha_5+\Bigg(-\frac{1}{3024}\,t\sim^{13}+\frac{1747}{15120}\,t\sim^{9}
-\frac{23}{80}\,t\sim^{5}-\frac{1}{360}\,t\sim^{11}-\frac{35}{16}\,t\sim-\frac{13}{8}\,t\sim^{3}+\frac{103}{420}\,t\sim^{7}\Bigg)\,\alpha_7+\Bigg(\frac{19}{3402}\,t\sim^{9}+\frac{1}{38880}\,t\sim^{11}
-\frac{23}{72}\,t\sim+\frac{59}{3024}\,t\sim^{7}-\frac{19}{216}\,t\sim^{5}-\frac{275}{864}\,t\sim^{3}\Bigg)\,\alpha_9\Bigg)\,\alpha_3+\Bigg(\frac{539}{324}\,t\sim^{9}+\frac{1}{583200}\,t\sim^{19}
-\frac{7}{324}\,t\sim^{13}-\frac{49}{36}\,t\sim^{5}-\frac{41}{48600}\,t\sim^{15}+\frac{175}{24}\,t\sim^{7}-\frac{679}{18}\,t\sim-\frac{10759}{288}\,t\sim^{3}+\frac{1}{24300}\,t\sim^{17}
-\frac{49}{6480}\,t\sim^{11}\Bigg)\,\alpha_5\,\alpha_3^{5}+\Bigg(\Bigg(-\frac{1}{209952}\,t\sim^{19}-\frac{1}{5184}\,t\sim^{17}-\frac{175}{192}\,t\sim^{5}-\frac{89425}{23328}\,t\sim^{9}
+\frac{606725}{7776}\,t\sim^{3}+\frac{423395}{5184}\,t\sim+\frac{35}{432}\,t\sim^{11}+\frac{1}{6298560}\,t\sim^{21}+\frac{31}{17496}\,t\sim^{15}-\frac{11375}{648}\,t\sim^{7}
+\frac{1435}{23328}\,t\sim^{13}\Bigg)\,\alpha_4-\frac{t\sim^{21}}{2099520}+\frac{649\,t\sim^{17}}{2099520}-\frac{847\,t\sim^{15}}{262440}+\frac{112315\,t\sim^{9}}{69984}-\frac{54565\,t\sim^{3}}{7776}
+\frac{1505\,t\sim^{11}}{34992}+\frac{19285\,t\sim^{5}}{15552}-\frac{21805\,t\sim}{1728}-\frac{3703\,t\sim^{13}}{69984}+\frac{13\,t\sim^{19}}{1049760}+\frac{6895\,t\sim^{7}}{1944}\Bigg)\,\alpha_3^{6}
+\frac{11693\,t\sim^{5}}{10240}-\frac{3821\,t\sim^{3}}{6144}+\frac{49709\,t\sim^{9}}{92160}-\frac{2317\,t\sim^{11}}{18432}-\frac{33383\,t\sim^{7}}{30720}+\frac{19\,t\sim^{13}}{2048}-\frac{t\sim^{15}}{6144}
+\Bigg(-\frac{2}{3}\,t\sim-\frac{3}{20}\,t\sim^{5}-\frac{31}{48}\,t\sim^{3}+\frac{121}{2520}\,t\sim^{7}+\frac{31}{3780}\,t\sim^{9}+\frac{1}{5040}\,t\sim^{11}\Bigg)\,\alpha_7\,\alpha_5+\Bigg(\Bigg(
-\frac{112}{9}\,t\sim^{7}-\frac{10087}{3888}\,t\sim^{9}+\frac{4361}{54}\,t\sim^{3}+\frac{497}{72}\,t\sim^{5}+\frac{2625}{32}\,t\sim+\frac{77}{3240}\,t\sim^{13}+\frac{7}{162}\,t\sim^{11}
+\frac{1}{3645}\,t\sim^{15}-\frac{1}{38880}\,t\sim^{17}\Bigg)\,\alpha_5\,\alpha_4+\Bigg(-\frac{29}{810}\,t\sim^{13}-\frac{1}{1296}\,t\sim^{15}+\frac{2303}{1296}\,t\sim^{9}-\frac{1}{9}\,t\sim^{5}
+\frac{1}{12960}\,t\sim^{17}-\frac{651}{32}\,t\sim-\frac{1843}{144}\,t\sim^{3}+\frac{127}{6480}\,t\sim^{11}+\frac{577}{144}\,t\sim^{7}\Bigg)\,\alpha_5+\Bigg(-\frac{373}{72}\,t\sim
+\frac{323}{1944}\,t\sim^{9}-\frac{11}{9720}\,t\sim^{13}-\frac{7}{8}\,t\sim^{5}-\frac{1169}{216}\,t\sim^{3}+\frac{1}{1080}\,t\sim^{11}+\frac{145}{216}\,t\sim^{7}
-\frac{1}{40824}\,t\sim^{15}\Bigg)\,\alpha_7\Bigg)\,\alpha_3^{3}+\Bigg(\Bigg(-\frac{5915}{864}\,t\sim^{7}+\frac{581}{7776}\,t\sim^{11}-\frac{37835}{31104}\,t\sim^{9}-\frac{11}{23328}\,t\sim^{15}
+\frac{133}{15552}\,t\sim^{13}+\frac{127225}{2304}\,t\sim+\frac{1}{186624}\,t\sim^{17}+\frac{44135}{864}\,t\sim^{3}+\frac{9065}{1728}\,t\sim^{5}\Bigg)\,\alpha_4^{3}+\Bigg(
-\frac{1}{20736}\,t\sim^{17}-\frac{167}{3456}\,t\sim^{13}+\frac{425}{64}\,t\sim^{7}-\frac{595}{24}\,t\sim^{3}+\frac{4685}{1728}\,t\sim^{9}+\frac{17}{5184}\,t\sim^{15}-\frac{10115}{256}\,t\sim
$$

$$
- \frac{133}{864}\,t\sim^{11} - \frac{135}{128}\,t\sim^{5}\Bigg)\,\alpha_4^2 + \Bigg(\Bigg(-\frac{805}{32}\,t\sim - \frac{49}{12960}\,t\sim^{13} + \frac{1505}{2592}\,t\sim^{9} - \frac{7}{432}\,t\sim^{11}
$$

$$
- \frac{119}{32}\,t\sim^{5} + \frac{1}{9720}\,t\sim^{15} - \frac{3493}{144}\,t\sim^{3} + \frac{49}{18}\,t\sim^{7}\Bigg)\,\alpha_6 - \frac{11\,t\sim^{15}}{1296} - \frac{5927\,t\sim^{9}}{3456} - \frac{37\,t\sim^{11}}{288}
$$

$$
+ \frac{1711\,t\sim}{256} - \frac{119\,t\sim^{5}}{96} + \frac{25\,t\sim^{13}}{216} + \frac{11\,t\sim^{7}}{36} + \frac{47\,t\sim^{3}}{32} + \frac{t\sim^{17}}{6912}\Bigg)\,\alpha_4 + \Bigg(-\frac{91}{43200}\,t\sim^{13}
$$

$$
+ \frac{1519}{960}\,t\sim^{7} - \frac{3843}{320}\,t\sim - \frac{1}{14400}\,t\sim^{15} - \frac{1393}{960}\,t\sim^{5} + \frac{581}{1728}\,t\sim^{9} + \frac{49}{43200}\,t\sim^{11}
$$

$$
- \frac{11669}{960}\,t\sim^{3}\Bigg)\,\alpha_5^2 + \Bigg(-\frac{1}{3240}\,t\sim^{15} + \frac{113}{12960}\,t\sim^{13} + \frac{187}{288}\,t\sim^{5} - \frac{1411}{2592}\,t\sim^{9} + \frac{259}{32}\,t\sim
$$

$$
+ \frac{791}{144}\,t\sim^{3} + \frac{47}{6480}\,t\sim^{11} - \frac{41}{36}\,t\sim^{7}\Bigg)\,\alpha_6 + \Bigg(-\frac{1}{4320}\,t\sim^{11} + \frac{67}{192}\,t\sim^{5} + \frac{151}{96}\,t\sim^{3}
$$

$$
+ \frac{11}{60480}\,t\sim^{13} - \frac{443}{12096}\,t\sim^{9} - \frac{15}{112}\,t\sim^{7} + \frac{99}{64}\,t\sim\Bigg)\,\alpha_8 - \frac{9\,t\sim}{256} + \frac{47\,t\sim^{11}}{108} + \frac{647\,t\sim^{5}}{1152}
$$

$$
- \frac{31\,t\sim^{3}}{96} - \frac{223\,t\sim^{9}}{576} + \frac{5\,t\sim^{15}}{576} - \frac{79\,t\sim^{7}}{576} - \frac{425\,t\sim^{13}}{3456} - \frac{t\sim^{17}}{6912}\Bigg)\,\alpha_3^2 + \Bigg(\Bigg(-\frac{2905}{576}\,t\sim^{5}
$$

$$
+ \frac{1}{15552}\,t\sim^{15} + \frac{35245}{1728}\,t\sim^{7} - \frac{791}{15552}\,t\sim^{13} - \frac{260995}{2304}\,t\sim^{3} + \frac{42875}{10368}\,t\sim^{9} + \frac{7}{62208}\,t\sim^{17}
$$

$$
- \frac{1477}{10368}\,t\sim^{11} - \frac{818825}{6912}\,t\sim - \frac{1}{559872}\,t\sim^{19}\Bigg)\,\alpha_4^2 + \Bigg(-\frac{69685}{15552}\,t\sim^{9} - \frac{4445}{432}\,t\sim^{7}
$$

$$
+ \frac{17395}{384}\,t\sim - \frac{17}{31104}\,t\sim^{17} + \frac{92855}{3456}\,t\sim^{3} + \frac{1}{93312}\,t\sim^{19} + \frac{7}{5832}\,t\sim^{15} + \frac{77}{5184}\,t\sim^{11}
$$

$$
+ \frac{119}{972}\,t\sim^{13} - \frac{245}{144}\,t\sim^{5}\Bigg)\,\alpha_4 + \Bigg(\frac{91}{14580}\,t\sim^{13} + \frac{5159}{324}\,t\sim^{3} - \frac{791}{324}\,t\sim^{7} + \frac{3409}{216}\,t\sim
$$

$$
- \frac{1}{87480}\,t\sim^{17} + \frac{1}{8748}\,t\sim^{15} + \frac{49}{14580}\,t\sim^{11} - \frac{847}{1458}\,t\sim^{9} + \frac{553}{324}\,t\sim^{5}\Bigg)\,\alpha_6 - \frac{2065\,t\sim}{768}
$$

$$
+ \frac{25345\,t\sim^{9}}{31104} + \frac{2083\,t\sim^{11}}{10368} + \frac{445\,t\sim^{5}}{576} - \frac{2785\,t\sim^{3}}{6912} + \frac{149\,t\sim^{17}}{186624} - \frac{215\,t\sim^{15}}{46656} - \frac{65\,t\sim^{7}}{1728}
$$

$$
- \frac{125\,t\sim^{13}}{1728} - \frac{t\sim^{19}}{62208}\Bigg)\,\alpha_3^4
$$

Check with Chung's method:

Q_tilde[1],...,Q_tilde[8] obtained with Chung's method

```
> Q_tilde[1]:=((1/3)*alpha[3]*t^2+(1/6)*alpha[3]):
```

```
> Q_tilde[2]:=-(1/18)*alpha[3]^2*t^5+((1/12)*alpha[4]-1/2-(1/9)*
```

```
alpha[3]^2)*t^3+(-(1/4)*alpha[4]+(1/6)*alpha[3]^2+1/2)*t+(1/4)*
t^3+(1/4)*t:


> Q_tilde[3]:=(1/162)*alpha[3]^3*t^8+((7/324)*alpha[3]^3+(1/6)*
alpha[3]-(1/36)*alpha[3]*alpha[4])*t^6+(-(5/12)*alpha[3]-(1/20)*
alpha[5]+(5/24)*alpha[3]*alpha[4]-(35/216)*alpha[3]^3)*t^4+(-
(175/432)*alpha[3]^3+(1/8)*alpha[3]+(5/8)*alpha[3]*alpha[4]-
(1/5)*alpha[5])*t^2-(1/16)*alpha[3]-(1/40)*alpha[5]-(35/432)*
alpha[3]^3+(5/48)*alpha[3]*alpha[4]+((1/2)*t^2-(1/4)*t^4+1/4)*(
(1/3)*alpha[3]*t^2+(1/6)*alpha[3]):

> Q_tilde[4]:=(-(5/1944)*t^9-(5/216)*t^3+(5/108)*t^7-(1/1944)*t^11
-(35/72)*t+(25/108)*t^5)*alpha[3]^4+(((1/216)*t^9-(1/18)*t^7-
(5/12)*t^5+(29/24)*t+(1/6)*t^3)*alpha[4]-(1/36)*t^9+(1/9)*t^7+
(2/9)*t^5-(1/12)*t-(2/9)*t^3+((1/2)*t^2-(1/4)*t^4+1/4)*(-(1/18)*
t^5+(1/6)*t-(1/9)*t^3))*alpha[3]^2+((2/15)*t^5+(1/60)*t^7-(1/12)
*t^3-(1/2)*t)*alpha[5]*alpha[3]+(-(37/96)*t-(1/288)*t^7+(7/96)*
t^5-(11/96)*t^3)*alpha[4]^2+((1/8)*t+(7/24)*t^3+(1/24)*t^7-
(7/24)*t^5+((1/2)*t^2-(1/4)*t^4+1/4)*(-(1/4)*t+(1/12)*t^3))*
alpha[4]+(-(1/45)*t^5+(1/18)*t^3+(1/6)*t)*alpha[6]-(65/96)*t^3+
(19/32)*t-(3/32)*t^7+(53/96)*t^5+((1/2)*t^2-(1/4)*t^4+1/4)*(-
(1/4)*t^3+(3/4)*t):

> Q_tilde[5] := (1/29160)*alpha[3]^5*t^14+(-(1/1944)*alpha[3]^3*
alpha[4]+(13/58320)*alpha[3]^5+(1/648)*alpha[3]^3)*t^12+(-
(143/19440)*alpha[3]^5-(25/1296)*alpha[3]^3-(1/360)*alpha[5]*
alpha[3]^2+(1/864)*alpha[3]*alpha[4]^2+(11/1296)*alpha[3]^3*
alpha[4]+(1/96)*alpha[3]-(1/144)*alpha[3]*alpha[4])*t^10+(-
(1/80)*alpha[5]-(235/2592)*alpha[3]^3-(109/576)*alpha[3]-(7/192)
*alpha[3]*alpha[4]^2+(1/240)*alpha[5]*alpha[4]+(13/96)*alpha[3]*
alpha[4]+(1/135)*alpha[3]*alpha[6]-(143/2592)*alpha[3]^5-(1/30)*
alpha[5]*alpha[3]^2+(11/96)*alpha[3]^3*alpha[4])*t^8+((149/288)*
alpha[3]-(43/144)*alpha[3]*alpha[4]+(1309/5184)*alpha[3]^3+
(1/20)*alpha[5]+(1/252)*alpha[7]+(49/480)*alpha[5]*alpha[3]^2-
(385/1728)*alpha[3]^3*alpha[4]-(7/360)*alpha[5]*alpha[4]+
(1001/10368)*alpha[3]^5+(7/96)*alpha[3]*alpha[4]^2-(7/216)*alpha
[3]*alpha[6])*t^6+((35/108)*alpha[3]^3-(25/48)*alpha[3]*alpha[4]
-(35/64)*alpha[3]+(7/40)*alpha[5]+(7007/6912)*alpha[3]^5-(7/16)*
alpha[5]*alpha[4]-(385/144)*alpha[3]^3*alpha[4]-(49/144)*alpha
[3]*alpha[6]+(35/32)*alpha[5]*alpha[3]^2+(11/168)*alpha[7]+
(245/192)*alpha[3]*alpha[4]^2)*t^4+((1001/864)*alpha[3]^5+
(875/3456)*alpha[3]^3-(25/64)*alpha[3]*alpha[4]+(161/384)*alpha
```

```
[3]+(3/56)*alpha[7]-(7/16)*alpha[5]*alpha[4]-(49/144)*alpha[3]*
alpha[6]+(77/64)*alpha[5]*alpha[3]^2-(385/128)*alpha[3]^3*alpha
[4]+(175/128)*alpha[3]*alpha[4]^2+(1/8)*alpha[5])*t^2+(49/768)*
alpha[3]+(13/320)*alpha[5]+(1/336)*alpha[7]-(7/288)*alpha[3]*
alpha[6]+(7/64)*alpha[5]*alpha[3]^2-(65/384)*alpha[3]*alpha[4]+
(455/3456)*alpha[3]^3-(7/192)*alpha[5]*alpha[4]+(35/256)*alpha
[3]*alpha[4]^2-(385/1152)*alpha[3]^3*alpha[4]+(1001/6912)*alpha
[3]^5:

> Q_tilde[6] := -(1/524880)*t^17*alpha[3]^6+(-(1/7776)*alpha[3]^4+
(1/23328)*alpha[3]^4*alpha[4]-(1/65610)*alpha[3]^6)*t^15+(
(17/7776)*alpha[3]^4-(1/576)*alpha[3]^2+(1/864)*alpha[3]^2*alpha
[4]-(1/5184)*alpha[3]^2*alpha[4]^2+(1/3240)*alpha[3]^3*alpha[5]-
(7/7776)*alpha[3]^4*alpha[4]+(7/8748)*alpha[3]^6)*t^13+(
(1/10368)*alpha[4]^3+(1/384)*alpha[4]-(49/2592)*alpha[3]^4*alpha
[4]+(43/864)*alpha[3]^2-(5/144)*alpha[3]^2*alpha[4]+(1/240)*
alpha[3]*alpha[5]-(1/1152)*alpha[4]^2+(169/7776)*alpha[3]^4
-1/384+(2/405)*alpha[3]^3*alpha[5]+(35/4374)*alpha[3]^6+(7/864)*
alpha[3]^2*alpha[4]^2-(1/810)*alpha[3]^2*alpha[6]-(1/720)*alpha
[3]*alpha[4]*alpha[5])*t^11+(-(55/10368)*alpha[4]^3-(77/1152)*
alpha[4]-(1/180)*alpha[6]-(1/800)*alpha[5]^2+(595/7776)*alpha[3]
^4*alpha[4]+9/128-(365/1728)*alpha[3]^2+(85/864)*alpha[3]^2*
alpha[4]-(1/40)*alpha[3]*alpha[5]+(35/1152)*alpha[4]^2-
(845/7776)*alpha[3]^4-(23/648)*alpha[3]^3*alpha[5]-(665/17496)*
alpha[3]^6-(1/756)*alpha[3]*alpha[7]-(25/1728)*alpha[3]^2*alpha
[4]^2+(5/648)*alpha[3]^2*alpha[6]+(1/540)*alpha[4]*alpha[6]+
(1/108)*alpha[3]*alpha[4]*alpha[5])*t^9+((35/576)*alpha[4]^3+
(21/64)*alpha[4]+(5/72)*alpha[6]+(11/3360)*alpha[8]-(13/600)*
alpha[5]^2+(1225/864)*alpha[3]^4*alpha[4]+(1/48)*alpha[3]^2+
(25/24)*alpha[3]^2*alpha[4]-(41/120)*alpha[3]*alpha[5]-(41/192)*
alpha[4]^2-(475/864)*alpha[3]^4-65/192-(5/9)*alpha[3]^3*alpha[5]
-(245/486)*alpha[3]^6-(11/315)*alpha[3]*alpha[7]-(125/144)*alpha
[3]^2*alpha[4]^2+(1/6)*alpha[3]^2*alpha[6]-(7/180)*alpha[4]*
alpha[6]+(3/8)*alpha[3]*alpha[4]*alpha[5])*t^7+((35/576)*alpha
[4]^3-(27/64)*alpha[4]-(13/360)*alpha[6]-(1/480)*alpha[8]-
(47/1200)*alpha[5]^2+(665/288)*alpha[3]^4*alpha[4]+(133/576)*
alpha[3]^2+(5/32)*alpha[3]^2*alpha[4]-(1/30)*alpha[3]*alpha[5]+
(11/192)*alpha[4]^2-(125/864)*alpha[3]^4+119/192-(61/72)*alpha
[3]^3*alpha[5]-(805/972)*alpha[3]^6-(1/30)*alpha[3]*alpha[7]-
(265/192)*alpha[3]^2*alpha[4]^2+(7/36)*alpha[3]^2*alpha[6]-
(1/60)*alpha[4]*alpha[6]+(7/12)*alpha[3]*alpha[4]*alpha[5])*t^5+
(-(835/1152)*alpha[4]^3+(31/128)*alpha[4]-(11/72)*alpha[6]-
```

```
(7/96)*alpha[8]+(29/120)*alpha[5]^2-(4585/864)*alpha[3]^4*alpha
[4]-(25/96)*alpha[3]^2-(13/16)*alpha[3]^2*alpha[4]+(17/48)*alpha
[3]*alpha[5]+(43/128)*alpha[4]^2+(235/864)*alpha[3]^4-185/384+
(23/9)*alpha[3]^3*alpha[5]+(665/486)*alpha[3]^6+(1/3)*alpha[3]*
alpha[7]+(475/96)*alpha[3]^2*alpha[4]^2-(10/9)*alpha[3]^2*alpha
[6]+(7/12)*alpha[4]*alpha[6]-(45/16)*alpha[3]*alpha[4]*alpha[5])
*t^3+(-(425/384)*alpha[4]^3-(33/128)*alpha[4]-(3/8)*alpha[6]-
(3/32)*alpha[8]+(51/160)*alpha[5]^2-(2485/288)*alpha[3]^4*alpha
[4]+(11/64)*alpha[3]^2-(87/32)*alpha[3]^2*alpha[4]+(9/8)*alpha
[3]*alpha[5]+105/128+(111/128)*alpha[4]^2+(35/32)*alpha[3]^4+
(31/8)*alpha[3]^3*alpha[5]+(3115/1296)*alpha[3]^6+(5/12)*alpha
[3]*alpha[7]+(1445/192)*alpha[3]^2*alpha[4]^2-(37/24)*alpha[3]
^2*alpha[6]+(5/6)*alpha[4]*alpha[6]-4*alpha[3]*alpha[4]*alpha[5]
)*t:
> Q_tilde[7] :=(39/2048)*alpha[3]-(12635/55296)*alpha[3]^3-
(11011/18432)*alpha[3]^5-(230945/497664)*alpha[3]^7+(1/11022480)
*alpha[3]^7*t^20-(231/512)*alpha[5]*alpha[3]^2+(85085/55296)*
alpha[3]^5*alpha[4]-(77/1024)*alpha[5]*alpha[4]^2-(1155/2048)*
alpha[3]*alpha[4]^2-(361/5120)*alpha[5]-(11/896)*alpha[7]-
(1/3456)*alpha[9]-(77/768)*alpha[3]*alpha[4]*alpha[6]+
(1001/1536)*alpha[5]*alpha[3]^2*alpha[4]-(77/1280)*alpha[3]*
alpha[5]^2+(77/512)*alpha[5]*alpha[4]-(25025/18432)*alpha[3]^3*
alpha[4]^2+(1/256)*alpha[3]*alpha[8]+(4235/3072)*alpha[3]^3*
alpha[4]+(5005/18432)*alpha[3]*alpha[4]^3+(7/640)*alpha[5]*alpha
[6]+(77/768)*alpha[3]*alpha[6]+(1001/6912)*alpha[3]^3*alpha[6]+
(1/128)*alpha[4]*alpha[7]-(11/384)*alpha[3]^2*alpha[7]-
(5005/9216)*alpha[5]*alpha[3]^4+(1805/6144)*alpha[3]*alpha[4]+(
(19/22044960)*alpha[3]^7-(1/349920)*alpha[3]^5*alpha[4]+
(1/116640)*alpha[3]^5)*t^18+((17/233280)*alpha[3]^5*alpha[4]+
(1/46656)*alpha[3]^3*alpha[4]^2-(1/38880)*alpha[5]*alpha[3]^4-
(43/233280)*alpha[3]^5+(1/5184)*alpha[3]^3-(323/4898880)*alpha
[3]^7-(1/7776)*alpha[3]^3*alpha[4])*t^16+(-(235/31104)*alpha[3]
^3-(1615/1959552)*alpha[3]^7+(1/1152)*alpha[3]+(1/7290)*alpha[3]
^3*alpha[6]-(1/1944)*alpha[5]*alpha[3]^4-(35/31104)*alpha[3]^3*
alpha[4]^2-(1/1440)*alpha[5]*alpha[3]^2-(1/31104)*alpha[3]*alpha
[4]^3-(359/116640)*alpha[3]^5+(1/3456)*alpha[3]*alpha[4]^2+
(1/192)*alpha[3]^3*alpha[4]+(1/4320)*alpha[5]*alpha[3]^2*alpha
[4]+(17/7776)*alpha[3]^5*alpha[4]-(1/1152)*alpha[3]*alpha[4])*
t^14+(-(221/15552)*alpha[3]^5*alpha[4]-(13/6480)*alpha[5]*alpha
[3]^2*alpha[4]+(143/62208)*alpha[3]*alpha[4]^3-(1/1620)*alpha[3]
*alpha[4]*alpha[6]-(29/768)*alpha[3]-(17/1152)*alpha[3]^3*alpha
[4]+(91/3888)*alpha[3]^5+(1/540)*alpha[3]*alpha[6]-(1/640)*alpha
```

```
[5]+(2489/62208)*alpha[3]^3+(1/960)*alpha[5]*alpha[4]+
(4199/559872)*alpha[3]^7+(1/4536)*alpha[3]^2*alpha[7]+(1/2400)*
alpha[3]*alpha[5]^2-(13/11664)*alpha[3]^3*alpha[6]+(65/62208)*
alpha[3]^3*alpha[4]^2-(1/5760)*alpha[5]*alpha[4]^2+(13/1944)*
alpha[5]*alpha[3]^4+(1/180)*alpha[5]*alpha[3]^2+(253/6912)*alpha
[3]*alpha[4]-(103/6912)*alpha[3]*alpha[4]^2)*t^12+((821/2304)*
alpha[3]+(7751/41472)*alpha[3]^3+(148291/622080)*alpha[3]^5+
(46189/373248)*alpha[3]^7+(901/5760)*alpha[5]*alpha[3]^2-
(46189/124416)*alpha[3]^5*alpha[4]+(11/2160)*alpha[5]*alpha[4]
^2+(1141/6912)*alpha[3]*alpha[4]^2+(17/480)*alpha[5]+(1/1008)*
alpha[7]+(77/4320)*alpha[3]*alpha[4]*alpha[6]-(143/1152)*alpha
[5]*alpha[3]^2*alpha[4]+(143/14400)*alpha[3]*alpha[5]^2-
(31/1440)*alpha[5]*alpha[4]+(33605/124416)*alpha[3]^3*alpha[4]^2
-(11/10080)*alpha[3]*alpha[8]-(10505/20736)*alpha[3]^3*alpha[4]-
(715/20736)*alpha[3]*alpha[4]^3-(1/900)*alpha[5]*alpha[6]-
(163/4320)*alpha[3]*alpha[6]-(1573/38880)*alpha[3]^3*alpha[6]-
(1/3024)*alpha[4]*alpha[7]+(121/15120)*alpha[3]^2*alpha[7]+
(715/5184)*alpha[5]*alpha[3]^4-(793/2304)*alpha[3]*alpha[4])*
t^10+(-(4627/4608)*alpha[3]-(6407/13824)*alpha[3]^3-(2431/82944)
*alpha[3]^5+(46189/248832)*alpha[3]^7-(11/384)*alpha[5]*alpha[3]
^2-(17017/27648)*alpha[3]^5*alpha[4]+(143/3840)*alpha[5]*alpha
[4]^2+(161/1536)*alpha[3]*alpha[4]^2-(49/768)*alpha[5]+(1/224)*
alpha[7]-(1/12960)*alpha[9]+(11/320)*alpha[3]*alpha[4]*alpha[6]-
(143/640)*alpha[5]*alpha[3]^2*alpha[4]+(11/960)*alpha[3]*alpha
[5]^2-(33/640)*alpha[5]*alpha[4]+(7865/13824)*alpha[3]^3*alpha
[4]^2+(1/960)*alpha[3]*alpha[8]-(11/768)*alpha[3]^3*alpha[4]-
(1859/13824)*alpha[3]*alpha[4]^3-(1/480)*alpha[5]*alpha[6]+
(97/8640)*alpha[3]*alpha[6]-(1001/25920)*alpha[3]^3*alpha[6]-
(13/3360)*alpha[4]*alpha[7]+(11/3360)*alpha[3]^2*alpha[7]+
(2431/13824)*alpha[5]*alpha[3]^4+(899/1536)*alpha[3]*alpha[4])*
t^8+((5987/4608)*alpha[3]+(35/2592)*alpha[3]^3-(17017/9216)*
alpha[3]^5-(1062347/497664)*alpha[3]^7-(1281/640)*alpha[5]*alpha
[3]^2+(221221/27648)*alpha[3]^5*alpha[4]-(77/120)*alpha[5]*alpha
[4]^2-(3521/1536)*alpha[3]*alpha[4]^2-(59/480)*alpha[5]-(41/336)
*alpha[7]-(19/1620)*alpha[9]-(77/72)*alpha[3]*alpha[4]*alpha[6]+
(7007/1440)*alpha[5]*alpha[3]^2*alpha[4]-(539/960)*alpha[3]*
alpha[5]^2+(63/80)*alpha[5]*alpha[4]-(55055/6912)*alpha[3]^3*
alpha[4]^2+(43/480)*alpha[3]*alpha[8]+(11165/2304)*alpha[3]^3*
alpha[4]+(25025/13824)*alpha[3]*alpha[4]^3+(7/48)*alpha[5]*alpha
[6]+(91/144)*alpha[3]*alpha[6]+(71071/51840)*alpha[3]^3*alpha[6]
+(31/240)*alpha[4]*alpha[7]-(1177/2880)*alpha[3]^2*alpha[7]-
(25025/6912)*alpha[5]*alpha[3]^4+(565/4608)*alpha[3]*alpha[4])*
```

```
t^6+(-(2807/3072)*alpha[3]-(7525/27648)*alpha[3]^3-
(181181/55296)*alpha[3]^5-(1154725/165888)*alpha[3]^7-(7/2)*
alpha[5]*alpha[3]^2+(1446445/55296)*alpha[3]^5*alpha[4]-
(3157/1536)*alpha[5]*alpha[4]^2-(12425/3072)*alpha[3]*alpha[4]^2
-(143/2560)*alpha[5]-(43/224)*alpha[7]-(23/864)*alpha[9]-(77/24)
*alpha[3]*alpha[4]*alpha[6]+(1001/64)*alpha[5]*alpha[3]^2*alpha
[4]-(231/128)*alpha[3]*alpha[5]^2+(1043/768)*alpha[5]*alpha[4]-
(725725/27648)*alpha[3]^3*alpha[4]^2+(15/64)*alpha[3]*alpha[8]+
(39655/4608)*alpha[3]^3*alpha[4]+(55055/9216)*alpha[3]*alpha[4]
^3+(7/16)*alpha[5]*alpha[6]+(203/192)*alpha[3]*alpha[6]+
(7007/1728)*alpha[3]^3*alpha[6]+(17/48)*alpha[4]*alpha[7]-
(55/48)*alpha[3]^2*alpha[7]-(35035/3072)*alpha[5]*alpha[3]^4+
(955/3072)*alpha[3]*alpha[4])*t^4+((1405/3072)*alpha[3]-
(49175/55296)*alpha[3]^3-(25025/6912)*alpha[3]^5-
(2540395/497664)*alpha[3]^7-(1925/512)*alpha[5]*alpha[3]^2+
(85085/4608)*alpha[3]^5*alpha[4]-(77/64)*alpha[5]*alpha[4]^2-
(4375/1024)*alpha[3]*alpha[4]^2-(281/640)*alpha[5]-(75/448)*
alpha[7]-(1/108)*alpha[9]-(231/128)*alpha[3]*alpha[4]*alpha[6]+
(5005/512)*alpha[5]*alpha[3]^2*alpha[4]-(1309/1280)*alpha[3]*
alpha[5]^2+(175/128)*alpha[5]*alpha[4]-(325325/18432)*alpha[3]
^3*alpha[4]^2+(13/128)*alpha[3]*alpha[8]+(9625/1024)*alpha[3]^3*
alpha[4]+(35035/9216)*alpha[3]*alpha[4]^3+(7/32)*alpha[5]*alpha
[6]+(1225/1152)*alpha[3]*alpha[6]+(17017/6912)*alpha[3]^3*alpha
[6]+(11/64)*alpha[4]*alpha[7]-(77/128)*alpha[3]^2*alpha[7]-
(35035/4608)*alpha[5]*alpha[3]^4+(1405/1024)*alpha[3]*alpha[4])*
t^2:
> Q_tilde[8] :=-(1/264539520)*alpha[3]^8*t^23+(-(1/2099520)*alpha
[3]^6+(1/6298560)*alpha[3]^6*alpha[4]-(11/264539520)*alpha[3]^8)
*t^21+(-(1/62208)*alpha[3]^4+(1/93312)*alpha[3]^4*alpha[4]+
(13/1049760)*alpha[3]^6+(11/2519424)*alpha[3]^8+(1/583200)*alpha
[5]*alpha[3]^5-(1/559872)*alpha[3]^4*alpha[4]^2-(1/209952)*alpha
[3]^6*alpha[4])*t^19+(-(1/20736)*alpha[3]^2*alpha[4]^2+(1/12960)
*alpha[3]^3*alpha[5]+(649/2099520)*alpha[3]^6+(1/6912)*alpha[3]
^2*alpha[4]-(17/31104)*alpha[3]^4*alpha[4]-(1/6912)*alpha[3]^2+
(7/62208)*alpha[3]^4*alpha[4]^2+(1/186624)*alpha[3]^2*alpha[4]^3
-(1/5184)*alpha[3]^6*alpha[4]-(1/38880)*alpha[3]^3*alpha[5]*
alpha[4]+(149/186624)*alpha[3]^4-(1/87480)*alpha[3]^4*alpha[6]+
(55/839808)*alpha[3]^8+(1/24300)*alpha[5]*alpha[3]^5)*t^17+(
(5/576)*alpha[3]^2-(1/9216)*alpha[4]^2-1/6144+(1/9720)*alpha[3]
^2*alpha[4]*alpha[6]+(1/41472)*alpha[4]^3+(1/8748)*alpha[3]^4*
alpha[6]-(11/23328)*alpha[3]^2*alpha[4]^3+(31/17496)*alpha[3]^6*
alpha[4]+(7/5832)*alpha[3]^4*alpha[4]-(41/48600)*alpha[5]*alpha
```

```
[3]^5+(1/15552)*alpha[3]^4*alpha[4]^2+(1/3645)*alpha[3]^3*alpha
[5]*alpha[4]-(1/2880)*alpha[3]*alpha[5]*alpha[4]+(1/17280)*alpha
[3]*alpha[5]*alpha[4]^2-(11/1296)*alpha[3]^2*alpha[4]+(1/4608)*
alpha[4]+(17/5184)*alpha[3]^2*alpha[4]^2-(407/419904)*alpha[3]^8
-(1/1296)*alpha[3]^3*alpha[5]-(847/262440)*alpha[3]^6-(1/40824)*
alpha[3]^3*alpha[7]-(1/14400)*alpha[3]^2*alpha[5]^2-(1/3240)*
alpha[3]^2*alpha[6]-(215/46656)*alpha[3]^4-(1/497664)*alpha[4]
^4+(1/1920)*alpha[3]*alpha[5])*t^15+(-(425/3456)*alpha[3]^2+
(179/27648)*alpha[4]^2-(1/3200)*alpha[5]^2-(49/12960)*alpha[3]
^2*alpha[4]*alpha[6]-(77/41472)*alpha[4]^3+(91/14580)*alpha[3]
^4*alpha[6]+(133/15552)*alpha[3]^2*alpha[4]^3+(1435/23328)*alpha
[3]^6*alpha[4]+(119/972)*alpha[3]^4*alpha[4]-(7/324)*alpha[5]*
alpha[3]^5-(791/15552)*alpha[3]^4*alpha[4]^2+19/2048+(1/9072)*
alpha[3]*alpha[4]*alpha[7]+(77/3240)*alpha[3]^3*alpha[5]*alpha
[4]+(43/4320)*alpha[3]*alpha[5]*alpha[4]-(7/3240)*alpha[3]*alpha
[5]*alpha[4]^2+(25/216)*alpha[3]^2*alpha[4]+(1/2700)*alpha[3]*
alpha[5]*alpha[6]-(53/4608)*alpha[4]-(1/1440)*alpha[6]-
(167/3456)*alpha[3]^2*alpha[4]^2-(1/12960)*alpha[4]^2*alpha[6]-
(2695/139968)*alpha[3]^8+(11/60480)*alpha[3]^2*alpha[8]+(1/2160)
*alpha[4]*alpha[6]+(1/9600)*alpha[4]*alpha[5]^2-(1/3024)*alpha
[3]*alpha[7]-(29/810)*alpha[3]^3*alpha[5]-(3703/69984)*alpha[3]
^6-(11/9720)*alpha[3]^3*alpha[7]-(91/43200)*alpha[3]^2*alpha[5]
^2+(113/12960)*alpha[3]^2*alpha[6]-(125/1728)*alpha[3]^4+
(35/165888)*alpha[4]^4-(23/1440)*alpha[3]*alpha[5])*t^13+(
(47/108)*alpha[3]^2-(317/3072)*alpha[4]^2+(1/4800)*alpha[5]^2-
(7/432)*alpha[3]^2*alpha[4]*alpha[6]+(1619/41472)*alpha[4]^3+
(49/14580)*alpha[3]^4*alpha[6]+(581/7776)*alpha[3]^2*alpha[4]^3+
(35/432)*alpha[3]^6*alpha[4]+(77/5184)*alpha[3]^4*alpha[4]-
(49/6480)*alpha[5]*alpha[3]^5-(1477/10368)*alpha[3]^4*alpha[4]^2
-2317/18432+(13/7560)*alpha[3]*alpha[4]*alpha[7]+(7/162)*alpha
[3]^3*alpha[5]*alpha[4]+(151/2880)*alpha[3]*alpha[5]*alpha[4]-
(427/17280)*alpha[3]*alpha[5]*alpha[4]^2-(37/288)*alpha[3]^2*
alpha[4]+(1/1080)*alpha[3]*alpha[5]*alpha[6]+(229/1536)*alpha[4]
+(199/8640)*alpha[6]+(11/13440)*alpha[8]-(133/864)*alpha[3]^2*
alpha[4]^2+(37/8640)*alpha[4]^2*alpha[6]-(1/4050)*alpha[6]^2-
(1925/139968)*alpha[3]^8-(1/4320)*alpha[3]^2*alpha[8]-(13/720)*
alpha[4]*alpha[6]+(1/4800)*alpha[4]*alpha[5]^2-(1/360)*alpha[3]*
alpha[7]+(127/6480)*alpha[3]^3*alpha[5]+(1505/34992)*alpha[3]^6+
(1/38880)*alpha[3]*alpha[9]+(1/1080)*alpha[3]^3*alpha[7]+
(49/43200)*alpha[3]^2*alpha[5]^2-(11/40320)*alpha[4]*alpha[8]+
(1/5040)*alpha[7]*alpha[5]+(47/6480)*alpha[3]^2*alpha[6]+
(2083/10368)*alpha[3]^4-(343/55296)*alpha[4]^4+(71/5760)*alpha
```

```
[3]*alpha[5])*t^11+(-(223/576)*alpha[3]^2+(4067/9216)*alpha[4]
^2+(649/9600)*alpha[5]^2+(1505/2592)*alpha[3]^2*alpha[4]*alpha
[6]-(7915/41472)*alpha[4]^3-(847/1458)*alpha[3]^4*alpha[6]-
(37835/31104)*alpha[3]^2*alpha[4]^3-(89425/23328)*alpha[3]^6*
alpha[4]-(69685/15552)*alpha[3]^4*alpha[4]+(539/324)*alpha[5]*
alpha[3]^5+(42875/10368)*alpha[3]^4*alpha[4]^2+49709/92160-
(263/3024)*alpha[3]*alpha[4]*alpha[7]-(10087/3888)*alpha[3]^3*
alpha[5]*alpha[4]-(509/432)*alpha[3]*alpha[5]*alpha[4]+
(1043/1728)*alpha[3]*alpha[5]*alpha[4]^2-(5927/3456)*alpha[3]^2*
alpha[4]-(19/56700)*alpha[10]-(59/540)*alpha[3]*alpha[5]*alpha
[6]-(303/512)*alpha[4]-(1219/8640)*alpha[6]-(23/1920)*alpha[8]+
(4685/1728)*alpha[3]^2*alpha[4]^2-(73/1728)*alpha[4]^2*alpha[6]+
(7/1620)*alpha[6]^2+(140525/139968)*alpha[3]^8-(443/12096)*alpha
[3]^2*alpha[8]+(7/54)*alpha[4]*alpha[6]-(301/5760)*alpha[4]*
alpha[5]^2+(1747/15120)*alpha[3]*alpha[7]+(2303/1296)*alpha[3]
^3*alpha[5]+(112315/69984)*alpha[3]^6+(19/3402)*alpha[3]*alpha
[9]+(323/1944)*alpha[3]^3*alpha[7]+(581/1728)*alpha[3]^2*alpha
[5]^2+(139/24192)*alpha[4]*alpha[8]+(31/3780)*alpha[7]*alpha[5]-
(1411/2592)*alpha[3]^2*alpha[6]+(25345/31104)*alpha[3]^4+
(6545/165888)*alpha[4]^4+(1679/2880)*alpha[3]*alpha[5])*t^9+(-
(79/576)*alpha[3]^2-(1963/9216)*alpha[4]^2+(497/2400)*alpha[5]
^2+(49/18)*alpha[3]^2*alpha[4]*alpha[6]-(2165/4608)*alpha[4]^3-
(791/324)*alpha[3]^4*alpha[6]-(5915/864)*alpha[3]^2*alpha[4]^3-
(11375/648)*alpha[3]^6*alpha[4]-(4445/432)*alpha[3]^4*alpha[4]+
(175/24)*alpha[5]*alpha[3]^5+(35245/1728)*alpha[3]^4*alpha[4]^2
-33383/30720-(107/252)*alpha[3]*alpha[4]*alpha[7]-(112/9)*alpha
[3]^3*alpha[5]*alpha[4]-(577/192)*alpha[3]*alpha[5]*alpha[4]+
(4025/1152)*alpha[3]*alpha[5]*alpha[4]^2+(11/36)*alpha[3]^2*
alpha[4]-(13/12600)*alpha[10]-(31/60)*alpha[3]*alpha[5]*alpha[6]
+(1465/1536)*alpha[4]+(23/288)*alpha[6]-(137/6720)*alpha[8]+
(425/64)*alpha[3]^2*alpha[4]^2-(71/288)*alpha[4]^2*alpha[6]+
(7/360)*alpha[6]^2+(136675/31104)*alpha[3]^8-(15/112)*alpha[3]
^2*alpha[8]+(11/40)*alpha[4]*alpha[6]-(163/480)*alpha[4]*alpha
[5]^2+(103/420)*alpha[3]*alpha[7]+(577/144)*alpha[3]^3*alpha[5]+
(6895/1944)*alpha[3]^6+(59/3024)*alpha[3]*alpha[9]+(145/216)*
alpha[3]^3*alpha[7]+(1519/960)*alpha[3]^2*alpha[5]^2+(107/4032)*
alpha[4]*alpha[8]+(121/2520)*alpha[7]*alpha[5]-(41/36)*alpha[3]
^2*alpha[6]-(65/1728)*alpha[3]^4+(5215/18432)*alpha[4]^4-
(217/1920)*alpha[3]*alpha[5])*t^7+((647/1152)*alpha[3]^2+
(401/3072)*alpha[4]^2-(1493/9600)*alpha[5]^2-(119/32)*alpha[3]
^2*alpha[4]*alpha[6]+(2825/4608)*alpha[4]^3+(553/324)*alpha[3]
^4*alpha[6]+(9065/1728)*alpha[3]^2*alpha[4]^3-(175/192)*alpha[3]
```

```
^6*alpha[4]-(245/144)*alpha[3]^4*alpha[4]-(49/36)*alpha[5]*alpha
[3]^5-(2905/576)*alpha[3]^4*alpha[4]^2+11693/10240+(47/48)*alpha
[3]*alpha[4]*alpha[7]+(497/72)*alpha[3]^3*alpha[5]*alpha[4]+
(127/96)*alpha[3]*alpha[5]*alpha[4]-(413/96)*alpha[3]*alpha[5]*
alpha[4]^2-(119/96)*alpha[3]^2*alpha[4]+(3/200)*alpha[10]+
(61/60)*alpha[3]*alpha[5]*alpha[6]-(423/512)*alpha[4]-(1/60)*
alpha[6]+(89/960)*alpha[8]-(135/128)*alpha[3]^2*alpha[4]^2+
(43/48)*alpha[4]^2*alpha[6]-(13/120)*alpha[6]^2+(105875/93312)*
alpha[3]^8+(67/192)*alpha[3]^2*alpha[8]-(49/80)*alpha[4]*alpha
[6]+(393/640)*alpha[4]*alpha[5]^2-(23/80)*alpha[3]*alpha[7]-
(1/9)*alpha[3]^3*alpha[5]+(19285/15552)*alpha[3]^6-(19/216)*
alpha[3]*alpha[9]-(7/8)*alpha[3]^3*alpha[7]-(1393/960)*alpha[3]
^2*alpha[5]^2-(11/64)*alpha[4]*alpha[8]-(3/20)*alpha[7]*alpha[5]
+(187/288)*alpha[3]^2*alpha[6]+(445/576)*alpha[3]^4-(4165/6144)*
alpha[4]^4+(7/20)*alpha[3]*alpha[5])*t^5+(-(31/96)*alpha[3]^2-
(749/1024)*alpha[4]^2-(1139/960)*alpha[5]^2-(3493/144)*alpha[3]
^2*alpha[4]*alpha[6]+(16745/4608)*alpha[4]^3+(5159/324)*alpha[3]
^4*alpha[6]+(44135/864)*alpha[3]^2*alpha[4]^3+(606725/7776)*
alpha[3]^6*alpha[4]+(92855/3456)*alpha[3]^4*alpha[4]-(10759/288)
*alpha[5]*alpha[3]^5-(260995/2304)*alpha[3]^4*alpha[4]^2
-3821/6144+(115/24)*alpha[3]*alpha[4]*alpha[7]+(4361/54)*alpha
[3]^3*alpha[5]*alpha[4]+(893/64)*alpha[3]*alpha[5]*alpha[4]-
(11921/384)*alpha[3]*alpha[5]*alpha[4]^2+(47/32)*alpha[3]^2*
alpha[4]+(17/360)*alpha[10]+(673/120)*alpha[3]*alpha[5]*alpha[6]
+(181/512)*alpha[4]+(197/576)*alpha[6]+(137/384)*alpha[8]-
(595/24)*alpha[3]^2*alpha[4]^2+(829/192)*alpha[4]^2*alpha[6]-
(11/24)*alpha[6]^2-(1510355/93312)*alpha[3]^8+(151/96)*alpha[3]
^2*alpha[8]-(139/48)*alpha[4]*alpha[6]+(1151/320)*alpha[4]*alpha
[5]^2-(13/8)*alpha[3]*alpha[7]-(1843/144)*alpha[3]^3*alpha[5]-
(54565/7776)*alpha[3]^6-(275/864)*alpha[3]*alpha[9]-(1169/216)*
alpha[3]^3*alpha[7]-(11669/960)*alpha[3]^2*alpha[5]^2-(85/128)*
alpha[4]*alpha[8]-(31/48)*alpha[7]*alpha[5]+(791/144)*alpha[3]
^2*alpha[6]-(2785/6912)*alpha[3]^4-(72695/18432)*alpha[4]^4-
(257/384)*alpha[3]*alpha[5])*t^3+(-(9/256)*alpha[3]^2-
(2183/1024)*alpha[4]^2-(1071/640)*alpha[5]^2-(805/32)*alpha[3]
^2*alpha[4]*alpha[6]+(2975/512)*alpha[4]^3+(3409/216)*alpha[3]
^4*alpha[6]+(127225/2304)*alpha[3]^2*alpha[4]^3+(423395/5184)*
alpha[3]^6*alpha[4]+(17395/384)*alpha[3]^4*alpha[4]-(679/18)*
alpha[5]*alpha[3]^5-(818825/6912)*alpha[3]^4*alpha[4]
^2+1659/2048+(79/16)*alpha[3]*alpha[4]*alpha[7]+(2625/32)*alpha
[3]^3*alpha[5]*alpha[4]+21*alpha[3]*alpha[5]*alpha[4]-(6349/192)
*alpha[3]*alpha[5]*alpha[4]^2+(1711/256)*alpha[3]^2*alpha[4]+
```

```
(7/120)*alpha[10]+(23/4)*alpha[3]*alpha[5]*alpha[6]+(27/512)*
alpha[4]+(59/64)*alpha[6]+(63/128)*alpha[8]-(10115/256)*alpha[3]
^2*alpha[4]^2+(341/64)*alpha[4]^2*alpha[6]-(67/120)*alpha[6]^2-
(535535/31104)*alpha[3]^8+(99/64)*alpha[3]^2*alpha[8]-(35/8)*
alpha[4]*alpha[6]+(2429/640)*alpha[4]*alpha[5]^2-(35/16)*alpha
[3]*alpha[7]-(651/32)*alpha[3]^3*alpha[5]-(21805/1728)*alpha[3]
^6-(23/72)*alpha[3]*alpha[9]-(373/72)*alpha[3]^3*alpha[7]-
(3843/320)*alpha[3]^2*alpha[5]^2-(311/384)*alpha[4]*alpha[8]-
(2/3)*alpha[7]*alpha[5]+(259/32)*alpha[3]^2*alpha[6]-(2065/768)*
alpha[3]^4-(30415/6144)*alpha[4]^4-(177/64)*alpha[3]*alpha[5])*
t:
```

```
> i:='i':
  for i from 1 to number_polynomials do:
    tt3:={seq(alpha[i3],i3=3..i+2)}:
    q1[i]:=collect(expand(Q_tilde[i]),tt3);
  end:

  for i from 1 to number_polynomials do;
    print();
    Q_tilde[i]:=collect(q1[i],tt3);
    print();
  end;
```

$$Q\_tilde_1 := \alpha_3 \left(\frac{1}{6} + \frac{t\sim^2}{3} \right) \tag{21}$$

$$Q\_tilde_2 := \left(-\frac{1}{18} t\sim^5 + \frac{1}{6} t\sim - \frac{1}{9} t\sim^3 \right) \alpha_3^2 + \left(-\frac{1}{4} t\sim + \frac{1}{12} t\sim^3 \right) \alpha_4 - \frac{t\sim^3}{4} + \frac{3\, t\sim}{4}$$

$$Q\_tilde_3 := \left(\frac{7}{324} t\sim^6 - \frac{35}{216} t\sim^4 + \frac{1}{162} t\sim^8 - \frac{35}{432} - \frac{175}{432} t\sim^2 \right) \alpha_3^3 + \left(\left(\frac{5}{48} + \frac{5}{8} t\sim^2 + \frac{5}{24} t\sim^4 - \frac{1}{36} t\sim^6 \right) \alpha_4 - \frac{7\, t\sim^4}{24} - \frac{1}{48} + \frac{t\sim^6}{12} + \frac{7\, t\sim^2}{24} \right) \alpha_3 + \left(-\frac{1}{40} - \frac{1}{5} t\sim^2 - \frac{1}{20} t\sim^4 \right) \alpha_5$$

$$Q\_tilde_4 := \left(-\frac{1}{1944} t\sim^{11} - \frac{35}{72} t\sim + \frac{5}{108} t\sim^7 - \frac{5}{1944} t\sim^9 + \frac{25}{108} t\sim^5 - \frac{5}{216} t\sim^3 \right) \alpha_3^4 + \left(\left(\frac{1}{6} t\sim^3 + \frac{1}{216} t\sim^9 + \frac{29}{24} t\sim - \frac{5}{12} t\sim^5 - \frac{1}{18} t\sim^7 \right) \alpha_4 - \frac{t\sim^3}{6} + \frac{t\sim^7}{9} + \frac{t\sim^5}{9} - \frac{t\sim}{24} - \frac{t\sim^9}{72} \right) \alpha_3^2 + \left(\frac{1}{60} t\sim^7 + \frac{2}{15} t\sim^5 - \frac{1}{2} t\sim - \frac{1}{12} t\sim^3 \right) \alpha_5 \alpha_3 + \left(-\frac{11}{96} t\sim^3 + \frac{7}{96} t\sim^5 \right.$$

$$-\frac{37}{96}\,t\sim-\frac{1}{288}\,t\sim^{7}\Bigr)\,\alpha_4^{2}+\Bigl(\frac{3}{16}\,t\sim^{3}+\frac{1}{16}\,t\sim+\frac{1}{48}\,t\sim^{7}-\frac{3}{16}\,t\sim^{5}\Bigr)\,\alpha_4+\Bigl(\frac{1}{18}\,t\sim^{3}$$
$$+\frac{1}{6}\,t\sim-\frac{1}{45}\,t\sim^{5}\Bigr)\,\alpha_6+\frac{25\,t\sim}{32}+\frac{23\,t\sim^{5}}{96}-\frac{35\,t\sim^{3}}{96}-\frac{t\sim^{7}}{32}$$

$$Q\_tilde_5:=\Bigl(\frac{7007}{6912}\,t\sim^{4}-\frac{143}{2592}\,t\sim^{8}+\frac{1001}{864}\,t\sim^{2}+\frac{1001}{6912}+\frac{1}{29160}\,t\sim^{14}+\frac{1001}{10368}\,t\sim^{6}$$
$$+\frac{13}{58320}\,t\sim^{12}-\frac{143}{19440}\,t\sim^{10}\Bigr)\,\alpha_3^{5}+\Bigl(\Bigl(-\frac{385}{144}\,t\sim^{4}+\frac{11}{96}\,t\sim^{8}+\frac{11}{1296}\,t\sim^{10}-\frac{385}{1152}$$
$$-\frac{1}{1944}\,t\sim^{12}-\frac{385}{1728}\,t\sim^{6}-\frac{385}{128}\,t\sim^{2}\Bigr)\,\alpha_4+\frac{875\,t\sim^{2}}{3456}+\frac{t\sim^{12}}{648}+\frac{455}{3456}-\frac{25\,t\sim^{10}}{1296}$$
$$-\frac{235\,t\sim^{8}}{2592}+\frac{35\,t\sim^{4}}{108}+\frac{1309\,t\sim^{6}}{5184}\Bigr)\,\alpha_3^{3}+\Bigl(-\frac{1}{360}\,t\sim^{10}+\frac{49}{480}\,t\sim^{6}+\frac{7}{64}+\frac{77}{64}\,t\sim^{2}$$
$$-\frac{1}{30}\,t\sim^{8}+\frac{35}{32}\,t\sim^{4}\Bigr)\,\alpha_5\,\alpha_3^{2}+\Bigl(\Bigl(\frac{1}{864}\,t\sim^{10}+\frac{7}{96}\,t\sim^{6}+\frac{245}{192}\,t\sim^{4}+\frac{175}{128}\,t\sim^{2}-\frac{7}{192}\,t\sim^{8}$$
$$+\frac{35}{256}\Bigr)\,\alpha_4^{2}+\Bigl(-\frac{25}{64}\,t\sim^{2}+\frac{13}{96}\,t\sim^{8}-\frac{25}{48}\,t\sim^{4}-\frac{65}{384}-\frac{1}{144}\,t\sim^{10}-\frac{43}{144}\,t\sim^{6}\Bigr)\,\alpha_4+\Bigl($$
$$-\frac{7}{288}+\frac{1}{135}\,t\sim^{8}-\frac{7}{216}\,t\sim^{6}-\frac{49}{144}\,t\sim^{4}-\frac{49}{144}\,t\sim^{2}\Bigr)\,\alpha_6+\frac{149\,t\sim^{6}}{288}+\frac{161\,t\sim^{2}}{384}$$
$$-\frac{35\,t\sim^{4}}{64}-\frac{109\,t\sim^{8}}{576}+\frac{49}{768}+\frac{t\sim^{10}}{96}\Bigr)\,\alpha_3+\Bigl(-\frac{7}{192}+\frac{1}{240}\,t\sim^{8}-\frac{7}{16}\,t\sim^{4}-\frac{7}{16}\,t\sim^{2}$$
$$-\frac{7}{360}\,t\sim^{6}\Bigr)\,\alpha_5\,\alpha_4+\Bigl(-\frac{1}{80}\,t\sim^{8}+\frac{1}{8}\,t\sim^{2}+\frac{1}{20}\,t\sim^{6}+\frac{13}{320}+\frac{7}{40}\,t\sim^{4}\Bigr)\,\alpha_5+\Bigl(\frac{1}{336}$$
$$+\frac{3}{56}\,t\sim^{2}+\frac{11}{168}\,t\sim^{4}+\frac{1}{252}\,t\sim^{6}\Bigr)\,\alpha_7$$

$$Q\_tilde_6:=\Bigl(\frac{7}{8748}\,t\sim^{13}-\frac{1}{65610}\,t\sim^{15}+\frac{35}{4374}\,t\sim^{11}-\frac{805}{972}\,t\sim^{5}-\frac{665}{17496}\,t\sim^{9}-\frac{1}{524880}\,t\sim^{17}$$
$$-\frac{245}{486}\,t\sim^{7}+\frac{3115}{1296}\,t\sim+\frac{665}{486}\,t\sim^{3}\Bigr)\,\alpha_3^{6}+\Bigl(\Bigl(-\frac{49}{2592}\,t\sim^{11}-\frac{7}{7776}\,t\sim^{13}+\frac{665}{288}\,t\sim^{5}$$
$$-\frac{4585}{864}\,t\sim^{3}+\frac{1}{23328}\,t\sim^{15}+\frac{1225}{864}\,t\sim^{7}+\frac{595}{7776}\,t\sim^{9}-\frac{2485}{288}\,t\sim\Bigr)\,\alpha_4+\frac{169\,t\sim^{11}}{7776}$$
$$-\frac{t\sim^{15}}{7776}-\frac{475\,t\sim^{7}}{864}+\frac{17\,t\sim^{13}}{7776}-\frac{125\,t\sim^{5}}{864}+\frac{235\,t\sim^{3}}{864}-\frac{845\,t\sim^{9}}{7776}+\frac{35\,t\sim}{32}\Bigr)\,\alpha_3^{4}+\Bigl($$
$$-\frac{61}{72}\,t\sim^{5}+\frac{23}{9}\,t\sim^{3}-\frac{23}{648}\,t\sim^{9}+\frac{2}{405}\,t\sim^{11}-\frac{5}{9}\,t\sim^{7}+\frac{31}{8}\,t\sim+\frac{1}{3240}\,t\sim^{13}\Bigr)\,\alpha_5\,\alpha_3^{3}+\Bigl(\Bigl($$
$$-\frac{265}{192}\,t\sim^{5}-\frac{125}{144}\,t\sim^{7}-\frac{25}{1728}\,t\sim^{9}+\frac{7}{864}\,t\sim^{11}-\frac{1}{5184}\,t\sim^{13}+\frac{475}{96}\,t\sim^{3}+\frac{1445}{192}\,t\sim\Bigr)\,\alpha_4^{2}$$
$$+\Bigl(\frac{1}{864}\,t\sim^{13}+\frac{25}{24}\,t\sim^{7}-\frac{13}{16}\,t\sim^{3}-\frac{5}{144}\,t\sim^{11}+\frac{5}{32}\,t\sim^{5}-\frac{87}{32}\,t\sim+\frac{85}{864}\,t\sim^{9}\Bigr)\,\alpha_4+\Bigl($$
$$-\frac{10}{9}\,t\sim^{3}+\frac{5}{648}\,t\sim^{9}+\frac{7}{36}\,t\sim^{5}-\frac{1}{810}\,t\sim^{11}+\frac{1}{6}\,t\sim^{7}-\frac{37}{24}\,t\sim\Bigr)\,\alpha_6+\frac{11\,t\sim}{64}+\frac{133\,t\sim^{5}}{576}$$
$$-\frac{25\,t\sim^{3}}{96}-\frac{365\,t\sim^{9}}{1728}+\frac{43\,t\sim^{11}}{864}+\frac{t\sim^{7}}{48}-\frac{t\sim^{13}}{576}\Bigr)\,\alpha_3^{2}+\Bigl(\Bigl(\frac{3}{8}\,t\sim^{7}+\frac{7}{12}\,t\sim^{5}-\frac{45}{16}\,t\sim^{3}$$
$$+\frac{1}{108}\,t\sim^{9}-4\,t\sim-\frac{1}{720}\,t\sim^{11}\Bigr)\,\alpha_5\,\alpha_4+\Bigl(-\frac{41}{120}\,t\sim^{7}+\frac{1}{240}\,t\sim^{11}-\frac{1}{30}\,t\sim^{5}-\frac{1}{40}\,t\sim^{9}$$

$$+\frac{9}{8}\,t\sim+\frac{17}{48}\,t\sim^{3}\Big)\,\alpha_5+\Big(\frac{5}{12}\,t\sim-\frac{11}{315}\,t\sim^{7}-\frac{1}{30}\,t\sim^{5}-\frac{1}{756}\,t\sim^{9}+\frac{1}{3}\,t\sim^{3}\Big)\,\alpha_7\Big)\,\alpha_3$$
$$+\Big(-\frac{425}{384}\,t\sim+\frac{1}{10368}\,t\sim^{11}-\frac{55}{10368}\,t\sim^{9}+\frac{35}{576}\,t\sim^{7}+\frac{35}{576}\,t\sim^{5}-\frac{835}{1152}\,t\sim^{3}\Big)\,\alpha_4^{3}+\Big($$
$$-\frac{1}{1152}\,t\sim^{11}+\frac{43}{128}\,t\sim^{3}+\frac{111}{128}\,t\sim-\frac{41}{192}\,t\sim^{7}+\frac{11}{192}\,t\sim^{5}+\frac{35}{1152}\,t\sim^{9}\Big)\,\alpha_4^{2}+\Big(\Big(\frac{5}{6}\,t\sim$$
$$-\frac{7}{180}\,t\sim^{7}+\frac{7}{12}\,t\sim^{3}+\frac{1}{540}\,t\sim^{9}-\frac{1}{60}\,t\sim^{5}\Big)\,\alpha_6+\frac{t\sim^{11}}{384}-\frac{27\,t\sim^{5}}{64}+\frac{21\,t\sim^{7}}{64}-\frac{77\,t\sim^{9}}{1152}$$
$$+\frac{31\,t\sim^{3}}{128}-\frac{33\,t\sim}{128}\Big)\,\alpha_4+\Big(-\frac{1}{800}\,t\sim^{9}-\frac{47}{1200}\,t\sim^{5}-\frac{13}{600}\,t\sim^{7}+\frac{51}{160}\,t\sim+\frac{29}{120}\,t\sim^{3}\Big)\,\alpha_5^{2}$$
$$+\Big(-\frac{11}{72}\,t\sim^{3}-\frac{1}{180}\,t\sim^{9}-\frac{13}{360}\,t\sim^{5}-\frac{3}{8}\,t\sim+\frac{5}{72}\,t\sim^{7}\Big)\,\alpha_6+\Big(-\frac{7}{96}\,t\sim^{3}+\frac{11}{3360}\,t\sim^{7}$$
$$-\frac{3}{32}\,t\sim-\frac{1}{480}\,t\sim^{5}\Big)\,\alpha_8+\frac{105\,t\sim}{128}+\frac{119\,t\sim^{5}}{192}-\frac{185\,t\sim^{3}}{384}+\frac{9\,t\sim^{9}}{128}-\frac{t\sim^{11}}{384}-\frac{65\,t\sim^{7}}{192}$$

$$Q\_tilde_7:=\Big(-\frac{1154725}{165888}\,t\sim^{4}-\frac{230945}{497664}+\frac{4199}{559872}\,t\sim^{12}-\frac{1062347}{497664}\,t\sim^{6}+\frac{46189}{373248}\,t\sim^{10}$$
$$+\frac{19}{22044960}\,t\sim^{18}-\frac{323}{4898880}\,t\sim^{16}+\frac{1}{11022480}\,t\sim^{20}-\frac{1615}{1959552}\,t\sim^{14}-\frac{2540395}{497664}\,t\sim^{2}$$
$$+\frac{46189}{248832}\,t\sim^{8}\Big)\,\alpha_3^{7}+\Big(\Big(-\frac{46189}{124416}\,t\sim^{10}+\frac{17}{7776}\,t\sim^{14}+\frac{85085}{55296}+\frac{85085}{4608}\,t\sim^{2}$$
$$-\frac{1}{349920}\,t\sim^{18}+\frac{221221}{27648}\,t\sim^{6}+\frac{17}{233280}\,t\sim^{16}+\frac{1446445}{55296}\,t\sim^{4}-\frac{17017}{27648}\,t\sim^{8}$$
$$-\frac{221}{15552}\,t\sim^{12}\Big)\,\alpha_4-\frac{25025\,t\sim^{2}}{6912}-\frac{2431\,t\sim^{8}}{82944}-\frac{181181\,t\sim^{4}}{55296}-\frac{359\,t\sim^{14}}{116640}-\frac{11011}{18432}$$
$$+\frac{91\,t\sim^{12}}{3888}+\frac{148291\,t\sim^{10}}{622080}-\frac{43\,t\sim^{16}}{233280}+\frac{t\sim^{18}}{116640}-\frac{17017\,t\sim^{6}}{9216}\Big)\,\alpha_3^{5}+\Big(-\frac{1}{1944}\,t\sim^{14}$$
$$-\frac{25025}{6912}\,t\sim^{6}+\frac{715}{5184}\,t\sim^{10}-\frac{1}{38880}\,t\sim^{16}-\frac{35035}{4608}\,t\sim^{2}-\frac{35035}{3072}\,t\sim^{4}+\frac{13}{1944}\,t\sim^{12}$$
$$-\frac{5005}{9216}+\frac{2431}{13824}\,t\sim^{8}\Big)\,\alpha_5\,\alpha_3^{4}+\Big(\Big(\frac{65}{62208}\,t\sim^{12}+\frac{33605}{124416}\,t\sim^{10}+\frac{7865}{13824}\,t\sim^{8}$$
$$-\frac{55055}{6912}\,t\sim^{6}-\frac{25025}{18432}-\frac{325325}{18432}\,t\sim^{2}+\frac{1}{46656}\,t\sim^{16}-\frac{35}{31104}\,t\sim^{14}-\frac{725725}{27648}\,t\sim^{4}\Big)\,\alpha_4^{2}$$
$$+\Big(-\frac{11}{768}\,t\sim^{8}+\frac{1}{192}\,t\sim^{14}-\frac{10505}{20736}\,t\sim^{10}+\frac{11165}{2304}\,t\sim^{6}-\frac{1}{7776}\,t\sim^{16}+\frac{9625}{1024}\,t\sim^{2}$$
$$+\frac{4235}{3072}-\frac{17}{1152}\,t\sim^{12}+\frac{39655}{4608}\,t\sim^{4}\Big)\,\alpha_4+\Big(\frac{17017}{6912}\,t\sim^{2}+\frac{1}{7290}\,t\sim^{14}+\frac{1001}{6912}$$
$$-\frac{1573}{38880}\,t\sim^{10}+\frac{7007}{1728}\,t\sim^{4}+\frac{71071}{51840}\,t\sim^{6}-\frac{13}{11664}\,t\sim^{12}-\frac{1001}{25920}\,t\sim^{8}\Big)\,\alpha_6-\frac{6407\,t\sim^{8}}{13824}$$
$$-\frac{49175\,t\sim^{2}}{55296}-\frac{12635}{55296}+\frac{t\sim^{16}}{5184}+\frac{7751\,t\sim^{10}}{41472}+\frac{2489\,t\sim^{12}}{62208}-\frac{235\,t\sim^{14}}{31104}-\frac{7525\,t\sim^{4}}{27648}$$
$$+\frac{35\,t\sim^{6}}{2592}\Big)\,\alpha_3^{3}+\Big(\Big(\frac{1001}{64}\,t\sim^{4}+\frac{7007}{1440}\,t\sim^{6}+\frac{5005}{512}\,t\sim^{2}-\frac{13}{6480}\,t\sim^{12}+\frac{1}{4320}\,t\sim^{14}$$
$$-\frac{143}{1152}\,t\sim^{10}-\frac{143}{640}\,t\sim^{8}+\frac{1001}{1536}\Big)\,\alpha_5\,\alpha_4+\Big(-\frac{1925}{512}\,t\sim^{2}+\frac{1}{180}\,t\sim^{12}-\frac{1281}{640}\,t\sim^{6}$$

$$
\begin{aligned}
&- \frac{1}{1440}\, t\sim^{14} - \frac{7}{2}\, t\sim^{4} + \frac{901}{5760}\, t\sim^{10} - \frac{231}{512} - \frac{11}{384}\, t\sim^{8}\Big)\, \alpha_5 + \Big(- \frac{1177}{2880}\, t\sim^{6} - \frac{77}{128}\, t\sim^{2} \\
&+ \frac{11}{3360}\, t\sim^{8} - \frac{11}{384} + \frac{121}{15120}\, t\sim^{10} - \frac{55}{48}\, t\sim^{4} + \frac{1}{4536}\, t\sim^{12}\Big)\, \alpha_7\Big)\, \alpha_3^2 + \Big(\Big(\frac{55055}{9216}\, t\sim^{4} \\
&- \frac{1859}{13824}\, t\sim^{8} - \frac{715}{20736}\, t\sim^{10} + \frac{35035}{9216}\, t\sim^{2} + \frac{5005}{18432} + \frac{25025}{13824}\, t\sim^{6} + \frac{143}{62208}\, t\sim^{12} \\
&- \frac{1}{31104}\, t\sim^{14}\Big)\, \alpha_4^3 + \Big(\frac{1}{3456}\, t\sim^{14} - \frac{12425}{3072}\, t\sim^{4} - \frac{103}{6912}\, t\sim^{12} + \frac{161}{1536}\, t\sim^{8} + \frac{1141}{6912}\, t\sim^{10} \\
&- \frac{1155}{2048} - \frac{4375}{1024}\, t\sim^{2} - \frac{3521}{1536}\, t\sim^{6}\Big)\, \alpha_4^2 + \Big(\Big(- \frac{77}{768} - \frac{231}{128}\, t\sim^{2} - \frac{77}{24}\, t\sim^{4} + \frac{11}{320}\, t\sim^{8} \\
&+ \frac{77}{4320}\, t\sim^{10} - \frac{77}{72}\, t\sim^{6} - \frac{1}{1620}\, t\sim^{12}\Big)\, \alpha_6 - \frac{793\, t\sim^{10}}{2304} + \frac{899\, t\sim^{8}}{1536} + \frac{253\, t\sim^{12}}{6912} \\
&+ \frac{1405\, t\sim^{2}}{1024} + \frac{955\, t\sim^{4}}{3072} + \frac{565\, t\sim^{6}}{4608} - \frac{t\sim^{14}}{1152} + \frac{1805}{6144}\Big)\, \alpha_4 + \Big(\frac{1}{2400}\, t\sim^{12} - \frac{77}{1280} \\
&- \frac{1309}{1280}\, t\sim^{2} - \frac{539}{960}\, t\sim^{6} - \frac{231}{128}\, t\sim^{4} + \frac{11}{960}\, t\sim^{8} + \frac{143}{14400}\, t\sim^{10}\Big)\, \alpha_5^2 + \Big(\frac{97}{8640}\, t\sim^{8} \\
&+ \frac{91}{144}\, t\sim^{6} + \frac{1225}{1152}\, t\sim^{2} + \frac{1}{540}\, t\sim^{12} + \frac{203}{192}\, t\sim^{4} + \frac{77}{768} - \frac{163}{4320}\, t\sim^{10}\Big)\, \alpha_6 + \Big(\frac{1}{960}\, t\sim^{8} \\
&+ \frac{43}{480}\, t\sim^{6} + \frac{15}{64}\, t\sim^{4} + \frac{13}{128}\, t\sim^{2} - \frac{11}{10080}\, t\sim^{10} + \frac{1}{256}\Big)\, \alpha_8 - \frac{2807\, t\sim^{4}}{3072} + \frac{39}{2048} \\
&- \frac{4627\, t\sim^{8}}{4608} + \frac{1405\, t\sim^{2}}{3072} + \frac{t\sim^{14}}{1152} + \frac{821\, t\sim^{10}}{2304} - \frac{29\, t\sim^{12}}{768} + \frac{5987\, t\sim^{6}}{4608}\Big)\, \alpha_3 \\
&+ \Big(\frac{143}{3840}\, t\sim^{8} - \frac{3157}{1536}\, t\sim^{4} - \frac{77}{64}\, t\sim^{2} - \frac{77}{120}\, t\sim^{6} + \frac{11}{2160}\, t\sim^{10} - \frac{1}{5760}\, t\sim^{12} \\
&- \frac{77}{1024}\Big)\, \alpha_5\, \alpha_4^2 + \Big(\Big(- \frac{33}{640}\, t\sim^{8} + \frac{1043}{768}\, t\sim^{4} + \frac{1}{960}\, t\sim^{12} + \frac{77}{512} + \frac{63}{80}\, t\sim^{6} \\
&- \frac{31}{1440}\, t\sim^{10} + \frac{175}{128}\, t\sim^{2}\Big)\, \alpha_5 + \Big(\frac{11}{64}\, t\sim^{2} - \frac{13}{3360}\, t\sim^{8} - \frac{1}{3024}\, t\sim^{10} + \frac{17}{48}\, t\sim^{4} + \frac{1}{128} \\
&+ \frac{31}{240}\, t\sim^{6}\Big)\, \alpha_7\Big)\, \alpha_4 + \Big(\Big(\frac{7}{640} - \frac{1}{900}\, t\sim^{10} + \frac{7}{32}\, t\sim^{2} + \frac{7}{16}\, t\sim^{4} + \frac{7}{48}\, t\sim^{6} \\
&- \frac{1}{480}\, t\sim^{8}\Big)\, \alpha_6 - \frac{49\, t\sim^{8}}{768} + \frac{17\, t\sim^{10}}{480} - \frac{361}{5120} - \frac{281\, t\sim^{2}}{640} - \frac{59\, t\sim^{6}}{480} - \frac{143\, t\sim^{4}}{2560} \\
&- \frac{t\sim^{12}}{640}\Big)\, \alpha_5 + \Big(\frac{1}{224}\, t\sim^{8} - \frac{75}{448}\, t\sim^{2} - \frac{11}{896} - \frac{43}{224}\, t\sim^{4} + \frac{1}{1008}\, t\sim^{10} - \frac{41}{336}\, t\sim^{6}\Big)\, \alpha_7 \\
&+ \Big(- \frac{1}{12960}\, t\sim^{8} - \frac{1}{3456} - \frac{1}{108}\, t\sim^{2} - \frac{23}{864}\, t\sim^{4} - \frac{19}{1620}\, t\sim^{6}\Big)\, \alpha_9
\end{aligned}
$$

$$
\begin{aligned}
Q\_tilde_8 := {} & \frac{1659\, t\sim}{2048} + \Big(- \frac{1139}{960}\, t\sim^{3} + \frac{649}{9600}\, t\sim^{9} + \frac{497}{2400}\, t\sim^{7} - \frac{1071}{640}\, t\sim - \frac{1493}{9600}\, t\sim^{5} \\
&+ \frac{1}{4800}\, t\sim^{11} - \frac{1}{3200}\, t\sim^{13}\Big)\, \alpha_5^2 + \Big(\Big(\frac{2429}{640}\, t\sim + \frac{393}{640}\, t\sim^{5} + \frac{1}{4800}\, t\sim^{11} + \frac{1151}{320}\, t\sim^{3} \\
&- \frac{163}{480}\, t\sim^{7} + \frac{1}{9600}\, t\sim^{13} - \frac{301}{5760}\, t\sim^{9}\Big)\, \alpha_5^2 + \Big(\frac{7}{54}\, t\sim^{9} + \frac{1}{2160}\, t\sim^{13} - \frac{49}{80}\, t\sim^{5} + \frac{11}{40}\, t\sim^{7} \\
&- \frac{35}{8}\, t\sim - \frac{139}{48}\, t\sim^{3} - \frac{13}{720}\, t\sim^{11}\Big)\, \alpha_6 + \Big(- \frac{11}{64}\, t\sim^{5} + \frac{107}{4032}\, t\sim^{7} - \frac{85}{128}\, t\sim^{3} - \frac{311}{384}\, t\sim
\end{aligned}
$$

$$+\frac{139}{24192}\,t\tilde{}^{9}-\frac{11}{40320}\,t\tilde{}^{11}\Big)\,\alpha_8+\frac{27\,t\tilde{}}{512}-\frac{423\,t\tilde{}^{5}}{512}+\frac{181\,t\tilde{}^{3}}{512}-\frac{303\,t\tilde{}^{9}}{512}+\frac{229\,t\tilde{}^{11}}{1536}$$
$$+\frac{1465\,t\tilde{}^{7}}{1536}-\frac{53\,t\tilde{}^{13}}{4608}+\frac{t\tilde{}^{15}}{4608}\Big)\,\alpha_4+\Big(\frac{136675}{31104}\,t\tilde{}^{7}-\frac{407}{419904}\,t\tilde{}^{15}-\frac{535535}{31104}\,t\tilde{}$$
$$+\frac{140525}{139968}\,t\tilde{}^{9}-\frac{1510355}{93312}\,t\tilde{}^{3}+\frac{11}{2519424}\,t\tilde{}^{19}-\frac{1925}{139968}\,t\tilde{}^{11}+\frac{55}{839808}\,t\tilde{}^{17}$$
$$-\frac{1}{264539520}\,t\tilde{}^{23}-\frac{2695}{139968}\,t\tilde{}^{13}-\frac{11}{264539520}\,t\tilde{}^{21}+\frac{105875}{93312}\,t\tilde{}^{5}\Big)\,\alpha_3^8+\Big(\Big($$
$$-\frac{71}{288}\,t\tilde{}^{7}-\frac{1}{12960}\,t\tilde{}^{13}+\frac{37}{8640}\,t\tilde{}^{11}+\frac{829}{192}\,t\tilde{}^{3}-\frac{73}{1728}\,t\tilde{}^{9}+\frac{43}{48}\,t\tilde{}^{5}+\frac{341}{64}\,t\tilde{}\Big)\,\alpha_6$$
$$+\frac{4067\,t\tilde{}^{9}}{9216}+\frac{401\,t\tilde{}^{5}}{3072}-\frac{1963\,t\tilde{}^{7}}{9216}+\frac{179\,t\tilde{}^{13}}{27648}-\frac{2183\,t\tilde{}}{1024}-\frac{749\,t\tilde{}^{3}}{1024}-\frac{t\tilde{}^{15}}{9216}$$
$$-\frac{317\,t\tilde{}^{11}}{3072}\Big)\,\alpha_4^2+\Big(-\frac{77}{41472}\,t\tilde{}^{13}-\frac{2165}{4608}\,t\tilde{}^{7}+\frac{1619}{41472}\,t\tilde{}^{11}+\frac{2975}{512}\,t\tilde{}+\frac{2825}{4608}\,t\tilde{}^{5}$$
$$+\frac{16745}{4608}\,t\tilde{}^{3}+\frac{1}{41472}\,t\tilde{}^{15}-\frac{7915}{41472}\,t\tilde{}^{9}\Big)\,\alpha_4^3+\Big(-\frac{4165}{6144}\,t\tilde{}^{5}+\frac{35}{165888}\,t\tilde{}^{13}$$
$$+\frac{5215}{18432}\,t\tilde{}^{7}+\frac{6545}{165888}\,t\tilde{}^{9}-\frac{343}{55296}\,t\tilde{}^{11}-\frac{72695}{18432}\,t\tilde{}^{3}-\frac{1}{497664}\,t\tilde{}^{15}-\frac{30415}{6144}\,t\tilde{}\Big)$$
$$\alpha_4^4+\Big(-\frac{11}{24}\,t\tilde{}^{3}+\frac{7}{360}\,t\tilde{}^{7}-\frac{1}{4050}\,t\tilde{}^{11}-\frac{67}{120}\,t\tilde{}-\frac{13}{120}\,t\tilde{}^{5}+\frac{7}{1620}\,t\tilde{}^{9}\Big)\,\alpha_6^2+\Big($$
$$-\frac{1219}{8640}\,t\tilde{}^{9}-\frac{1}{1440}\,t\tilde{}^{13}-\frac{1}{60}\,t\tilde{}^{5}+\frac{199}{8640}\,t\tilde{}^{11}+\frac{197}{576}\,t\tilde{}^{3}+\frac{59}{64}\,t\tilde{}+\frac{23}{288}\,t\tilde{}^{7}\Big)\,\alpha_6$$
$$+\Big(\frac{11}{13440}\,t\tilde{}^{11}-\frac{23}{1920}\,t\tilde{}^{9}+\frac{137}{384}\,t\tilde{}^{3}+\frac{63}{128}\,t\tilde{}-\frac{137}{6720}\,t\tilde{}^{7}+\frac{89}{960}\,t\tilde{}^{5}\Big)\,\alpha_8$$
$$+\Big(\frac{7}{120}\,t\tilde{}-\frac{13}{12600}\,t\tilde{}^{7}+\frac{17}{360}\,t\tilde{}^{3}+\frac{3}{200}\,t\tilde{}^{5}-\frac{19}{56700}\,t\tilde{}^{9}\Big)\,\alpha_{10}+\Big(\Big(-\frac{427}{17280}\,t\tilde{}^{11}$$
$$-\frac{6349}{192}\,t\tilde{}+\frac{1043}{1728}\,t\tilde{}^{9}+\frac{1}{17280}\,t\tilde{}^{15}-\frac{7}{3240}\,t\tilde{}^{13}+\frac{4025}{1152}\,t\tilde{}^{7}-\frac{11921}{384}\,t\tilde{}^{3}$$
$$-\frac{413}{96}\,t\tilde{}^{5}\Big)\,\alpha_5\,\alpha_4^2+\Big(\Big(-\frac{1}{2880}\,t\tilde{}^{15}+\frac{43}{4320}\,t\tilde{}^{13}+\frac{127}{96}\,t\tilde{}^{5}+\frac{893}{64}\,t\tilde{}^{3}+\frac{151}{2880}\,t\tilde{}^{11}$$
$$-\frac{509}{432}\,t\tilde{}^{9}+21\,t\tilde{}-\frac{577}{192}\,t\tilde{}^{7}\Big)\,\alpha_5+\Big(\frac{13}{7560}\,t\tilde{}^{11}-\frac{263}{3024}\,t\tilde{}^{9}+\frac{79}{16}\,t\tilde{}+\frac{47}{48}\,t\tilde{}^{5}$$
$$+\frac{1}{9072}\,t\tilde{}^{13}-\frac{107}{252}\,t\tilde{}^{7}+\frac{115}{24}\,t\tilde{}^{3}\Big)\,\alpha_7\Big)\,\alpha_4+\Big(\Big(-\frac{31}{60}\,t\tilde{}^{7}+\frac{1}{2700}\,t\tilde{}^{13}+\frac{673}{120}\,t\tilde{}^{3}$$
$$+\frac{61}{60}\,t\tilde{}^{5}-\frac{59}{540}\,t\tilde{}^{9}+\frac{23}{4}\,t\tilde{}+\frac{1}{1080}\,t\tilde{}^{11}\Big)\,\alpha_6-\frac{217\,t\tilde{}^{7}}{1920}-\frac{23\,t\tilde{}^{13}}{1440}-\frac{177\,t\tilde{}}{64}$$
$$+\frac{t\tilde{}^{15}}{1920}+\frac{1679\,t\tilde{}^{9}}{2880}+\frac{7\,t\tilde{}^{5}}{20}-\frac{257\,t\tilde{}^{3}}{384}+\frac{71\,t\tilde{}^{11}}{5760}\Big)\,\alpha_5+\Big(-\frac{1}{3024}\,t\tilde{}^{13}+\frac{1747}{15120}\,t\tilde{}^{9}$$
$$-\frac{23}{80}\,t\tilde{}^{5}-\frac{1}{360}\,t\tilde{}^{11}-\frac{35}{16}\,t\tilde{}-\frac{13}{8}\,t\tilde{}^{3}+\frac{103}{420}\,t\tilde{}^{7}\Big)\,\alpha_7+\Big(\frac{19}{3402}\,t\tilde{}^{9}+\frac{1}{38880}\,t\tilde{}^{11}$$
$$-\frac{23}{72}\,t\tilde{}+\frac{59}{3024}\,t\tilde{}^{7}-\frac{19}{216}\,t\tilde{}^{5}-\frac{275}{864}\,t\tilde{}^{3}\Big)\,\alpha_9\Big)\,\alpha_3+\Big(\frac{539}{324}\,t\tilde{}^{9}+\frac{1}{583200}\,t\tilde{}^{19}$$
$$-\frac{7}{324}\,t\tilde{}^{13}-\frac{49}{36}\,t\tilde{}^{5}-\frac{41}{48600}\,t\tilde{}^{15}+\frac{175}{24}\,t\tilde{}^{7}-\frac{679}{18}\,t\tilde{}-\frac{10759}{288}\,t\tilde{}^{3}+\frac{1}{24300}\,t\tilde{}^{17}$$

$$
-\frac{49}{6480}\,t\sim^{11}\Big)\,\alpha_5\,\alpha_3^5+\Big(\Big(-\frac{1}{209952}\,t\sim^{19}-\frac{1}{5184}\,t\sim^{17}-\frac{175}{192}\,t\sim^{5}-\frac{89425}{23328}\,t\sim^{9}
+\frac{606725}{7776}\,t\sim^{3}+\frac{423395}{5184}\,t\sim+\frac{35}{432}\,t\sim^{11}+\frac{1}{6298560}\,t\sim^{21}+\frac{31}{17496}\,t\sim^{15}-\frac{11375}{648}\,t\sim^{7}
+\frac{1435}{23328}\,t\sim^{13}\Big)\,\alpha_4-\frac{t\sim^{21}}{2099520}+\frac{649\,t\sim^{17}}{2099520}-\frac{847\,t\sim^{15}}{262440}+\frac{112315\,t\sim^{9}}{69984}-\frac{54565\,t\sim^{3}}{7776}
+\frac{1505\,t\sim^{11}}{34992}+\frac{19285\,t\sim^{5}}{15552}-\frac{21805\,t\sim}{1728}-\frac{3703\,t\sim^{13}}{69984}+\frac{13\,t\sim^{19}}{1049760}+\frac{6895\,t\sim^{7}}{1944}\Big)\,\alpha_3^6
+\frac{11693\,t\sim^{5}}{10240}-\frac{3821\,t\sim^{3}}{6144}+\frac{49709\,t\sim^{9}}{92160}-\frac{2317\,t\sim^{11}}{18432}-\frac{33383\,t\sim^{7}}{30720}+\frac{19\,t\sim^{13}}{2048}-\frac{t\sim^{15}}{6144}
+\Big(-\frac{2}{3}\,t\sim-\frac{3}{20}\,t\sim^{5}-\frac{31}{48}\,t\sim^{3}+\frac{121}{2520}\,t\sim^{7}+\frac{31}{3780}\,t\sim^{9}+\frac{1}{5040}\,t\sim^{11}\Big)\,\alpha_7\,\alpha_5+\Big(\Big(
-\frac{112}{9}\,t\sim^{7}-\frac{10087}{3888}\,t\sim^{9}+\frac{4361}{54}\,t\sim^{3}+\frac{497}{72}\,t\sim^{5}+\frac{2625}{32}\,t\sim+\frac{77}{3240}\,t\sim^{13}+\frac{7}{162}\,t\sim^{11}
+\frac{1}{3645}\,t\sim^{15}-\frac{1}{38880}\,t\sim^{17}\Big)\,\alpha_5\,\alpha_4+\Big(-\frac{29}{810}\,t\sim^{13}-\frac{1}{1296}\,t\sim^{15}+\frac{2303}{1296}\,t\sim^{9}-\frac{1}{9}\,t\sim^{5}
+\frac{1}{12960}\,t\sim^{17}-\frac{651}{32}\,t\sim-\frac{1843}{144}\,t\sim^{3}+\frac{127}{6480}\,t\sim^{11}+\frac{577}{144}\,t\sim^{7}\Big)\,\alpha_5+\Big(-\frac{373}{72}\,t\sim
+\frac{323}{1944}\,t\sim^{9}-\frac{11}{9720}\,t\sim^{13}-\frac{7}{8}\,t\sim^{5}-\frac{1169}{216}\,t\sim^{3}+\frac{1}{1080}\,t\sim^{11}+\frac{145}{216}\,t\sim^{7}
-\frac{1}{40824}\,t\sim^{15}\Big)\,\alpha_7\Big)\,\alpha_3^3+\Big(\Big(-\frac{5915}{864}\,t\sim^{7}+\frac{581}{7776}\,t\sim^{11}-\frac{37835}{31104}\,t\sim^{9}-\frac{11}{23328}\,t\sim^{15}
+\frac{133}{15552}\,t\sim^{13}+\frac{127225}{2304}\,t\sim+\frac{1}{186624}\,t\sim^{17}+\frac{44135}{864}\,t\sim^{3}+\frac{9065}{1728}\,t\sim^{5}\Big)\,\alpha_4^3+\Big(
-\frac{1}{20736}\,t\sim^{17}-\frac{167}{3456}\,t\sim^{13}+\frac{425}{64}\,t\sim^{7}-\frac{595}{24}\,t\sim^{3}+\frac{4685}{1728}\,t\sim^{9}+\frac{17}{5184}\,t\sim^{15}-\frac{10115}{256}\,t\sim
-\frac{133}{864}\,t\sim^{11}-\frac{135}{128}\,t\sim^{5}\Big)\,\alpha_4^2+\Big(\Big(-\frac{805}{32}\,t\sim-\frac{49}{12960}\,t\sim^{13}+\frac{1505}{2592}\,t\sim^{9}-\frac{7}{432}\,t\sim^{11}
-\frac{119}{32}\,t\sim^{5}+\frac{1}{9720}\,t\sim^{15}-\frac{3493}{144}\,t\sim^{3}+\frac{49}{18}\,t\sim^{7}\Big)\,\alpha_6-\frac{11\,t\sim^{15}}{1296}-\frac{5927\,t\sim^{9}}{3456}-\frac{37\,t\sim^{11}}{288}
+\frac{1711\,t\sim}{256}-\frac{119\,t\sim^{5}}{96}+\frac{25\,t\sim^{13}}{216}+\frac{11\,t\sim^{7}}{36}+\frac{47\,t\sim^{3}}{32}+\frac{t\sim^{17}}{6912}\Big)\,\alpha_4+\Big(-\frac{91}{43200}\,t\sim^{13}
+\frac{1519}{960}\,t\sim^{7}-\frac{3843}{320}\,t\sim-\frac{1}{14400}\,t\sim^{15}-\frac{1393}{960}\,t\sim^{5}+\frac{581}{1728}\,t\sim^{9}+\frac{49}{43200}\,t\sim^{11}
-\frac{11669}{960}\,t\sim^{3}\Big)\,\alpha_5^2+\Big(-\frac{1}{3240}\,t\sim^{15}+\frac{113}{12960}\,t\sim^{13}+\frac{187}{288}\,t\sim^{5}-\frac{1411}{2592}\,t\sim^{9}+\frac{259}{32}\,t\sim
+\frac{791}{144}\,t\sim^{3}+\frac{47}{6480}\,t\sim^{11}-\frac{41}{36}\,t\sim^{7}\Big)\,\alpha_6+\Big(-\frac{1}{4320}\,t\sim^{11}+\frac{67}{192}\,t\sim^{5}+\frac{151}{96}\,t\sim^{3}
+\frac{11}{60480}\,t\sim^{13}-\frac{443}{12096}\,t\sim^{9}-\frac{15}{112}\,t\sim^{7}+\frac{99}{64}\,t\sim\Big)\,\alpha_8-\frac{9\,t\sim}{256}+\frac{47\,t\sim^{11}}{108}+\frac{647\,t\sim^{5}}{1152}
-\frac{31\,t\sim^{3}}{96}-\frac{223\,t\sim^{9}}{576}+\frac{5\,t\sim^{15}}{576}-\frac{79\,t\sim^{7}}{576}-\frac{425\,t\sim^{13}}{3456}-\frac{t\sim^{17}}{6912}\Big)\,\alpha_3^2+\Big(\Big(-\frac{2905}{576}\,t\sim^{5}
+\frac{1}{15552}\,t\sim^{15}+\frac{35245}{1728}\,t\sim^{7}-\frac{791}{15552}\,t\sim^{13}-\frac{260995}{2304}\,t\sim^{3}+\frac{42875}{10368}\,t\sim^{9}+\frac{7}{62208}\,t\sim^{17}
$$

$$-\frac{1477}{10368}\,t\sim^{11}-\frac{818825}{6912}\,t\sim-\frac{1}{559872}\,t\sim^{19}\Bigg)\alpha_4^2+\Bigg(-\frac{69685}{15552}\,t\sim^{9}-\frac{4445}{432}\,t\sim^{7}$$
$$+\frac{17395}{384}\,t\sim-\frac{17}{31104}\,t\sim^{17}+\frac{92855}{3456}\,t\sim^{3}+\frac{1}{93312}\,t\sim^{19}+\frac{7}{5832}\,t\sim^{15}+\frac{77}{5184}\,t\sim^{11}$$
$$+\frac{119}{972}\,t\sim^{13}-\frac{245}{144}\,t\sim^{5}\Bigg)\alpha_4+\Bigg(\frac{91}{14580}\,t\sim^{13}+\frac{5159}{324}\,t\sim^{3}-\frac{791}{324}\,t\sim^{7}+\frac{3409}{216}\,t\sim$$
$$-\frac{1}{87480}\,t\sim^{17}+\frac{1}{8748}\,t\sim^{15}+\frac{49}{14580}\,t\sim^{11}-\frac{847}{1458}\,t\sim^{9}+\frac{553}{324}\,t\sim^{5}\Bigg)\alpha_6-\frac{2065\,t\sim}{768}$$
$$+\frac{25345\,t\sim^{9}}{31104}+\frac{2083\,t\sim^{11}}{10368}+\frac{445\,t\sim^{5}}{576}-\frac{2785\,t\sim^{3}}{6912}+\frac{149\,t\sim^{17}}{186624}-\frac{215\,t\sim^{15}}{46656}-\frac{65\,t\sim^{7}}{1728}$$
$$-\frac{125\,t\sim^{13}}{1728}-\frac{t\sim^{19}}{62208}\Bigg)\alpha_3^4$$

Check with Chung's method:

```
> for i from 1 to min(8,number_polynomials) do:
    difference[i]:=expand(Q_tilde[i]-PP_mom[i]):
    print(i,difference[i]);
  end:
```

$$1,\ 0$$
$$2,\ 0$$
$$3,\ 0$$
$$4,\ 0$$
$$5,\ 0$$
$$6,\ 0$$
$$7,\ 0$$
$$8,\ 0$$

(22)