Maple-Worksheet for Chung's (1946) method for computing Edgeworth polynomials for self-normalized sums

Authors: Helmut Finner and Thorsten Dickhaus.

Last updated: February 23, 2009

Students t-statistic commonly defined by the one-Sample t-statistic
with nu=n-1 degrees of freedom.

Note: Student's t-statistc in Chung (1946) defined with norming sequence 1/n
in the variance estimator.

Edgeworth expansion has the form:

F_n (t) = Phi(t) + \sum_{i=1}^m p_i (t) phi (t)

with

Phi: cdf of standard normal

phi: pdf of standard normal

p_i (t) depends on moments alpha[3], ... , alpha[i+2]
of the underlying universe

References:

Finner, H., Dickhaus, T. (2009).
EDGEWORTH EXPANSIONS AND RATES OF
CONVERGENCE FOR NORMALIZED SUMS:
CHUNG'S 1946 METHOD REVISITED.
Preprint.

Bartlett, M. S. (1935). The effect of non-normality on the t distribution. Proc. Camb.
Philos. Soc. 31, 223-231.
Chung, K.-L. (1946). The approximate distribution of Student's statistics. Ann.
Math. Stat. 17, 447-465.
Gayen, A. K. (1949). The distribution of 'Student's' t in random samples of any size
drawn from non-normal universes. Biometrika 36, 353-369.
Hall, P. (1987). Edgeworth expansion for Student's t statistic under minimal moment
conditions. Ann. Prob. 15, 920-931.
Hall, P. (1992). The bootstrap and Edgeworth expansion. Springer Series in Statistics,

New York.

Hsu, P. L. (1945). The approximate distributions of the mean and variance of a sample of independent variables. Ann. Math. Stat. 16, 1-29.
Wallace, D. L. (1958). Asymptotic approximations to distributions. Ann. Math. Stat. 29, 635-654.

```
> restart;
```

USER INPUT: number of approximation polynomials needed

(Number_polynomials $>=$ 9 may take a long time)

```
> number_polynomials:=8;
```

$$number\_polynomials := 8 \qquad \textbf{(1)}$$

Initialize variables

```
> n:='n':
  k_approx:='k_approx':
  p:='p':
  q:='q':
  rho:='rho':

  assume(p::nonnegint,q::integer,rho>-1,rho<1,n::posint);
  for run from 1 to number_polynomials+2 do;
  assume(alpha[run]::real);
  end;
  alpha[1]:=0;
  alpha[2]:=1;
  k_approx:=number_polynomials+3;
```

$$\alpha_1 := 0 \qquad \textbf{(2)}$$
$$\alpha_2 := 1$$
$$k\_approx := 11$$

Define function g as at the beginning of Section 2 in Finner & Dickhaus (2009):

```
> g:=(l,lambda,x,z)->z*(1+lambda^2*z^2)^(-1/2)*(1+sum(GAMMA(3/2)
  /GAMMA(3/2-j)/GAMMA(j+1)*(alpha[4]-1)^(j/2)*x^(j)*(lambda^j),j=
```

```
1..1));
```

$$g := (l, \lambda, x, z) \rightarrow \frac{z \left(1 + \sum_{j=1}^{l} \frac{\Gamma\left(\frac{3}{2}\right) (\alpha_4 - 1)^{\frac{j}{2}} x^j \lambda^j}{\Gamma\left(\frac{3}{2} - j\right) \Gamma(j+1)}\right)}{\sqrt{1 + \lambda^2 z^2}} \tag{3}$$

Construct formal derivatives for the function g in lambda=0:

```
> f_diff[1,lambda]:=diff(ggg(lambda),lambda)*www(x,ggg(lambda)):
  for i from 2 to number_polynomials do:
    f_diff[i,lambda]:=simplify(Diff(f_diff[1,lambda],lambda$(i-1))
  );
    #print(f_diff[i,lambda]);
  end:
> i:='i':
  for i from 1 to number_polynomials do:
    gdiff[i]:=simplify(unapply(diff(g(k_approx,lambda,x,z),
  lambda$i),(lambda,x,z))):
    print(gdiff[i](0, x, z));
  end:
```

$$\frac{1}{2} z \sqrt{\alpha_4 - 1}\, x \tag{4}$$

$$-z^3 - \frac{1}{4} z \left(\alpha_4 - 1\right) x^2$$

$$-\frac{3}{2} z^3 \sqrt{\alpha_4 - 1}\, x + \frac{3}{8} z \left(\alpha_4 - 1\right)^{3/2} x^3$$

$$9 z^5 + \frac{3}{2} z^3 \left(\alpha_4 - 1\right) x^2 - \frac{15}{16} z \left(\alpha_4 - 1\right)^2 x^4$$

$$\frac{45}{2} z^5 \sqrt{\alpha_4 - 1}\, x - \frac{15}{4} z^3 \left(\alpha_4 - 1\right)^{3/2} x^3 + \frac{105}{32} z \left(\alpha_4 - 1\right)^{5/2} x^5$$

$$-\frac{945}{64} z \left(\alpha_4 - 1\right)^3 x^6 - 225 z^7 - \frac{135}{4} z^5 \left(\alpha_4 - 1\right) x^2 + \frac{225}{16} z^3 \left(\alpha_4 - 1\right)^2 x^4$$

$$-\frac{1575}{2} z^7 \sqrt{\alpha_4 - 1}\, x + \frac{945}{8} z^5 \left(\alpha_4 - 1\right)^{3/2} x^3 + \frac{10395}{128} z \left(\alpha_4 - 1\right)^{7/2} x^7 - \frac{2205}{32} z^3 \left(\alpha_4 - 1\right)^{5/2} x^5$$

$$11025 z^9 - \frac{4725}{8} z^5 \left(\alpha_4 - 1\right)^2 x^4 + 1575 z^7 \left(\alpha_4 - 1\right) x^2 - \frac{135135}{256} z \left(\alpha_4 - 1\right)^4 x^8 + \frac{6615}{16} z^3 \left(\alpha_4 - 1\right)^3 x^6$$

Compute the f_pq^j's (see Lemma 2.2 in Finner & Dickhaus (2009)):

```
> ii1:='ii1':
  for ii1 from 1 to number_polynomials do:
    ww2:=f_diff[ii1,lambda];
    for j from number_polynomials to 1 by -1 do:
      ww3:=subs(diff(ggg(lambda),`$`(lambda,j))='gdiff[j](0,x,z)',
  ww2);
      ww2:=ww3;
      ww3:=subs(eval(diff(www(x,t1),`$`(t1,j)),{t1 = ggg(lambda)})
  =zzzz^(j+1),ww2);
      ww2:=ww3;
    end:
    ww3:=subs(www(x,ggg(lambda))=zzzz,ww2):
    ww2:=(expand(ww3));
    ww2;
    i:='i';
    j:='j';
    y:='y';
    zzzz:='zzzz';
    for i from 3*number_polynomials to 0 by -1 do;
      for j from 3*number_polynomials to 1 by -1 do;
        ww3:=algsubs(x^i*zzzz^j=II[p,q+j-1,i],ww2);
        ww2:=ww3;
      end;
    end;
    #print(ww2);
    fff[ii1]:=unapply(ww2,(p,q));
  end:
```

Compute Taylor approximation of f_pq (Formula 2.4 in Finner & Dickhaus (2009)):

```
> i:='i':
  pp:='pp':
  qq:='qq':
  iz1:='iz1':
  fff_all:=unapply(sum(fff[iz1](pp,qq)*lambda^iz1/iz1!,iz1=1..
  number_polynomials),(pp,qq)):
```

Definition of function w and its derivatives (top of page 4 in Finner & Dickhaus (2009)):

```
> y:='y';
  assume(rho>0,rho<1);
```

```
w:=(x,y,rho)->exp(-(x^2-2*rho*x*y+y^2)/(2*(1-rho^2)))/(2*Pi*sqrt
(1-rho^2));
```

$$y := y \tag{5}$$

$$w := (x, y, \rho) \rightarrow \frac{1}{2} \frac{e^{-\dfrac{x^2 - 2\rho xy + y^2}{2 - 2\rho^2}}}{\pi \sqrt{1 - \rho^2}}$$

```
> w_pq:=(x,y,rho,p,q)->diff(w(x,y,rho),x$p,y$q);
```

$$w\_pq := (x, y, \rho, p, q) \rightarrow \frac{\partial^q}{\partial y^q \, \partial x^p} \, w(x, y, \rho) \tag{6}$$

Definition of I_pq's:

```
> I_pq:=(y,rho,p,q,r)->int(x^r*w_pq(x,y,rho,p,q),x=-infinity..
  infinity);
  I_pq(y,rho,0,0,0):=Phi(y):
```

$$I\_pq := (y, \rho, p, q, r) \rightarrow \int_{-\infty}^{\infty} x^r \, w\_pq(x, y, \rho, p, q) \, dx \tag{7}$$

cdf and pdf of standard normal and its derivatives:

```
> phi:=x->exp(-x^2/2)/sqrt(2*Pi);
  Phi:=x->int(phi(y),y=-infinity..x);
  poly :=unapply((diff(phi(x),x$n) / phi(x)),(x, n));
```

$$\phi := x \rightarrow \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} \tag{8}$$

$$\Phi := x \rightarrow \int_{-\infty}^{x} \phi(y) \, dy$$

$$poly := (x, n\sim) \rightarrow \frac{\dfrac{d^{n\sim}}{dx^{n\sim}} e^{-\frac{1}{2}x^2}}{e^{-\frac{1}{2}x^2}}$$

Initialize integral values:

```
> II:='II':
  p:='p':
  q:='q':
```

```
   Phi:='Phi':
   rho := alpha[3] / sqrt(alpha[4]-1):
```

```
> iis := 'iis':
   for kk1 from 0 to 6*(number_polynomials) do
    for kk2 from -1 to 6*(number_polynomials) do
     for kk3 from 0 to 6*(number_polynomials) do
      II[kk1, kk2, kk3] := 0;
     end:
    end:
   end:
   i:='i':
   j:='j':
   q:='q':
   r:='r':
   II[0,0,0]:=Phi[0]:
```

Compute the remaining I_pq's via Lemma 2.1 in Finner & Dickhaus (2009):

```
> q:='q':
   r:='r':
   i:='i':
   j:='j':
   with(orthopoly):
   with(PolynomialTools):
   h:=(n,x)->expand((I/sqrt(2))^n*H(n,I*x/sqrt(2)));
   for q from 0 to 3*(number_polynomials) do;
     for r from 1 to number_polynomials do;
        II[0,q,r]:=sum(rho^i*coeff(h(r,x),x,i)*Phi[q+i],i=0..r);
     end;
   end;
   for q from 1 to 3*(number_polynomials) do:
     II[0,q,0]:=Phi[q];
   end:
   iis := 'iis':
   for q from 0 to 3*(number_polynomials) do;
     for p from 1 to number_polynomials do;
       for iis from p to number_polynomials do;
         II[p,q,iis]:=-iis*II[p-1,q,iis-1];
       end;
     end;
   end;
```

$$h := (n, x) \rightarrow expand\left(\left(\frac{I}{\sqrt{2}}\right)^{n} H\left(n, \frac{x\,I}{\sqrt{2}}\right)\right) \qquad\qquad \textbf{(9)}$$

Compute Psi(t1,t2) as sum s^j/j! according to formula (2.2) in Finner & Dickhaus (2009):

```
> r:='r':
  s := (k, t1, t2, lambda) -> I^3*lambda * sum(I^r*U[r+3](t1, t2)
  / ((r+3)! * lambda^(-r)), r=0..k-3);
```

$$s := (k, t1, t2, \lambda) \rightarrow -I\lambda\left(\sum_{r=0}^{k-3} \frac{I^{r} U_{r+3}(t1, t2)}{(r+3)!\,\lambda^{-r}}\right) \qquad\qquad \textbf{(10)}$$

```
> j := 'j':
  psi := (k, t1, t2, lambda) -> sum((s(k, t1, t2, lambda))^j / j!,
  j=1..(k-3));
```

$$\psi := (k, t1, t2, \lambda) \rightarrow \sum_{j=1}^{k-3} \frac{s(k, t1, t2, \lambda)^{j}}{j!} \qquad\qquad \textbf{(11)}$$

```
> rr0:=(t1,t2,lambda)->collect(expand(psi(k_approx, t1, t2,
  lambda)),lambda):
> rr_test:=(t1,t2,lambda)->convert(series(rr0(t1,t2,lambda),
  lambda=0, number_polynomials+1),polynom):
> rr:=unapply(rr_test(t1,t2,lambda),(t1,t2,lambda)):
```

Computation of cumulants U[j]:

```
> func := (t1, t2, zeta, alpha_1, k) -> (t1*(zeta^2-1)/sqrt
  (alpha_1-1)+t2*zeta)^k;
```

$$func := (t1, t2, \zeta, alpha\_1, k) \rightarrow \left(\frac{t1\,(\zeta^{2}-1)}{\sqrt{alpha\_1-1}} + t2\,\zeta\right)^{k} \qquad\qquad \textbf{(12)}$$

```
> zeta:='zeta':
  for i from 2 to number_polynomials+2 do:
    beta0:=expand(func(t1,t2,zeta,alpha[4],i));
    thelp1:=beta0;
    for j from 2*i to 1 by -1 do:
      thelp2:=subs(zeta^j=alpha[j],thelp1):
      thelp1:=thelp2;
      #print(thelp1,thelp2);
    end:
    beta[i]:=thelp1;
    #print(i,beta[i]);
```

```
   end:
   beta[1]:=0:
>  m:='m':
   n:='n':
   j:='j':
   t:='t':
   i_ende:=k_approx:
   kappa:='kappa':
   f1_moment:=(t,n)->log(1+sum(beta[j]/j!*(I*t)^j,j=1..
   (number_polynomials+2))):
   f1_moment(t,n):
   equation2:=unapply(convert(series(f1_moment(t,n),t=0,i_ende),
   polynom,t),t):
   for i from 1 to i_ende-1 do:
     kap1[i]:=expand(coeff(equation2(t),t^i)*i!/I^i);
   end:
>  for i from 3 to number_polynomials+2 do:
     U[i]:=unapply(kap1[i],(t1, t2));
   end:
```

Compute w_pq's:

```
>  p:='p':
   for p from 0 to 3*(number_polynomials) do;
     for q from 0 to 3*(number_polynomials) do;
       helpq:=fff_all(p,q);
       w[p,q]:=II[p,q-1,0]+helpq;
     end;
   end;
```

Plug cumulants in expansion for Psi
and substitute w_pq's in the expansion for Psi:

```
>  tt_new1:=expand(rr(t1,t2,lambda)):
```

```
>  tt_new2:=collect(tt_new1,[t1,t2],distributed):
   nops(expand(tt_new1));
```

$$142988 \tag{13}$$

```
>  t_vec:=[t1,t2]:
   C1:=[op(sort(tt_new2, t_vec, plex))]:
   num_C1:=nops(C1);
```

```
with(PolynomialTools);
i:='i';
tt:=0;
for i from 1 to num_C1 do:
   t1_exp:=degree(C1[i],t1):
   t2_exp:=degree(C1[i],t2):
   tt:=tt+algsubs((I*t1)^t1_exp*(I*t2)^t2_exp=(-1)^(t1_exp+
t2_exp)*w[t1_exp,t2_exp],C1[i]):
end:
```

$$num\_C1 := 295 \tag{14}$$

[CoefficientList, CoefficientVector, GcdFreeBasis, GreatestFactorialFactorization, Hurwitz, IsSelfReciprocal, MinimalPolynomial, PDEToPolynomial, PolynomialToPDE, ShiftEquivalent, ShiftlessDecomposition, Shorten, Shorter, Sort, Split, Splits, Translate]

$$i := i$$
$$tt := 0$$

Expand the resulting expression for Psi in powers of lambda:

```
> tt2:=collect(expand(tt),lambda):
> tt3:=collect(convert(series(tt2,lambda=0,k_approx-2),polynom),
  lambda):

> tt4:=(z,lambda)->tt3:
  tt5:=collect(collect(expand(tt4(z,lambda)),lambda),{lambda}):
> collect(expand(tt5),lambda):
> k := 'k':
  tt6a := algsubs(Phi[0] = 1, tt5):
  tt5:=tt6a:
  for k from 1 to 3*(number_polynomials) do:
    tt6a := algsubs(Phi[k] = poly(z,k), tt5):
    tt5:=tt6a:
  end:
  tt6:=convert(series(tt5,lambda=0,k_approx-2),polynom):
```

Final expansion of Psi:

```
> help1a := (unapply(tt6, (z, lambda))):
> help1b:=collect(expand(convert(series(simplify(expand(help1a(z,
  lambda))),lambda=0,k_approx-2),polynom)),lambda):
> help1:=unapply(help1b,(z,lambda)):
```

Expansion for f_00:

```
> bb1:= eval(fff_all(0,0)):
> bb2:=algsubs(Phi[0] = 1, bb1):
  for k from 1 to 3*(number_polynomials) do
    bb3 := algsubs(Phi[k] = poly(z,k), bb2);
    bb2:=bb3;
  end:
  bb4:=convert(series(bb2,lambda=0,k_approx-2),polynom):
> help2a := (unapply(bb4, (z, lambda))):
> help2b := collect(expand(convert(series(simplify(expand(help2a
  (z,lambda))),lambda=0,k_approx-2),polynom)),lambda):
  help2:=unapply(help2b,(z,lambda)):
```

Combine both expansions:

```
> pre_final:= simplify(help1(z,lambda)+help2(z,lambda)):
> final:=collect(expand(pre_final),lambda):
> PPP[1]:=limit(final/lambda,lambda=0);
  i:='i':
  j:='j':
  i1:='i1':
  for i from 2 to number_polynomials do:
    help1:=(final-sum(lambda^j*PPP[j],j=1..i-1))/lambda^i;
    PP1[i]:=expand(limit(help1,lambda=0));
    help2:={seq(alpha[i1],i1=3..i+2)};
    PPP[i]:=collect(PP1[i],help2);
  end:
```

$$PPP_1 := \frac{1}{6}\,\alpha_3 + \frac{1}{3}\,z^2\alpha_3 \qquad\qquad \textbf{(15)}$$

Polynomials in the Edgeworth expansion

```
> i2:='i2';
  for i2 from 1 to number_polynomials do;
    print( );
    PP[i2]:=PPP[i2];
    print( );
  end;
```

$$i2 := i2 \qquad\qquad \textbf{(16)}$$

$$PP_1 := \frac{1}{6}\,\alpha_3 + \frac{1}{3}\,z^2\alpha_3$$

$$PP_2 := \left(-\frac{1}{9}z^3 - \frac{1}{18}z^5 + \frac{1}{6}z\right)\alpha_3^2 + \left(-\frac{1}{4}z + \frac{1}{12}z^3\right)\alpha_4 - \frac{z^3}{2}$$

$$PP_3 := \left(\frac{1}{162}z^8 - \frac{35}{432} - \frac{35}{216}z^4 - \frac{175}{432}z^2 + \frac{7}{324}z^6\right)\alpha_3^3 + \left(\left(\frac{5}{8}z^2 - \frac{1}{36}z^6 + \frac{5}{24}z^4\right.\right.$$
$$\left.+ \frac{5}{48}\right)\alpha_4 - \frac{1}{16} - \frac{z^4}{4} + \frac{z^6}{6} - \frac{z^2}{8}\right)\alpha_3 + \left(-\frac{1}{40} - \frac{1}{20}z^4 - \frac{1}{5}z^2\right)\alpha_5$$

$$PP_4 := \left(\frac{25}{108}z^5 + \frac{5}{108}z^7 - \frac{5}{1944}z^9 - \frac{1}{1944}z^{11} - \frac{35}{72}z - \frac{5}{216}z^3\right)\alpha_3^4 + \left(\left(-\frac{1}{18}z^7\right.\right.$$
$$\left.+ \frac{1}{6}z^3 + \frac{29}{24}z - \frac{5}{12}z^5 + \frac{1}{216}z^9\right)\alpha_4 - \frac{z}{6} + \frac{z^7}{12} + \frac{z^3}{36} - \frac{z^9}{36} + \frac{11z^5}{36}\right)\alpha_3^2 + \left(\right.$$
$$\left.-\frac{1}{2}z - \frac{1}{12}z^3 + \frac{2}{15}z^5 + \frac{1}{60}z^7\right)\alpha_5\alpha_3 + \left(\frac{7}{96}z^5 - \frac{1}{288}z^7 - \frac{37}{96}z - \frac{11}{96}z^3\right)\alpha_4^2$$
$$+ \left(\frac{1}{4}z - \frac{1}{4}z^5 + \frac{1}{24}z^3 + \frac{1}{24}z^7\right)\alpha_4 + \left(\frac{1}{18}z^3 - \frac{1}{45}z^5 + \frac{1}{6}z\right)\alpha_6 + \frac{3z^5}{8} - \frac{z^7}{8}$$

$$PP_5 := \left(\frac{13}{58320}z^{12} + \frac{1}{29160}z^{14} + \frac{7007}{6912}z^4 + \frac{1001}{10368}z^6 - \frac{143}{19440}z^{10} - \frac{143}{2592}z^8 + \frac{1001}{6912}\right.$$
$$\left.+ \frac{1001}{864}z^2\right)\alpha_3^5 + \left(\left(-\frac{385}{1728}z^6 - \frac{1}{1944}z^{12} - \frac{385}{128}z^2 + \frac{11}{1296}z^{10} - \frac{385}{144}z^4 - \frac{385}{1152}\right.\right.$$
$$\left.+ \frac{11}{96}z^8\right)\alpha_4 + \frac{385z^4}{576} + \frac{35z^6}{432} + \frac{175}{1152} - \frac{z^{10}}{72} + \frac{875z^2}{1152} - \frac{17z^8}{108} + \frac{z^{12}}{324}\right)\alpha_3^3$$
$$+ \left(\frac{49}{480}z^6 + \frac{77}{64}z^2 + \frac{35}{32}z^4 - \frac{1}{30}z^8 + \frac{7}{64} - \frac{1}{360}z^{10}\right)\alpha_5\alpha_3^2 + \left(\left(\frac{35}{256} + \frac{7}{96}z^6\right.\right.$$
$$\left.- \frac{7}{192}z^8 + \frac{175}{128}z^2 + \frac{1}{864}z^{10} + \frac{245}{192}z^4\right)\alpha_4^2 + \left(\frac{3}{16}z^8 - \frac{185}{192}z^4 - \frac{75}{64}z^2 - \frac{1}{72}z^{10}\right.$$
$$\left.- \frac{5}{96}z^6 - \frac{25}{128}\right)\alpha_4 + \left(-\frac{49}{144}z^2 + \frac{1}{135}z^8 - \frac{49}{144}z^4 - \frac{7}{216}z^6 - \frac{7}{288}\right)\alpha_6 + \frac{25z^2}{384}$$
$$+ \frac{25}{768} + \frac{z^{10}}{24} + \frac{3z^6}{16} - \frac{5z^8}{16} + \frac{3z^4}{32}\right)\alpha_3 + \left(-\frac{7}{192} - \frac{7}{16}z^4 - \frac{7}{16}z^2 + \frac{1}{240}z^8\right.$$
$$\left.- \frac{7}{360}z^6\right)\alpha_5\alpha_4 + \left(\frac{3}{8}z^2 + \frac{3}{64} - \frac{1}{40}z^8 + \frac{9}{32}z^4\right)\alpha_5 + \left(\frac{1}{252}z^6 + \frac{3}{56}z^2 + \frac{11}{168}z^4\right.$$
$$\left.+ \frac{1}{336}\right)\alpha_7$$

$$PP_6 := \left(-\frac{245}{486}z^7 - \frac{1}{65610}z^{15} - \frac{665}{17496}z^9 + \frac{665}{486}z^3 + \frac{35}{4374}z^{11} + \frac{3115}{1296}z - \frac{1}{524880}z^{17}\right.$$
$$\left.+ \frac{7}{8748}z^{13} - \frac{805}{972}z^5\right)\alpha_3^6 + \left(\left(\frac{595}{7776}z^9 + \frac{665}{288}z^5 - \frac{7}{7776}z^{13} - \frac{49}{2592}z^{11} - \frac{2485}{288}z\right.\right.$$
$$\left.- \frac{4585}{864}z^3 + \frac{1225}{864}z^7 + \frac{1}{23328}z^{15}\right)\alpha_4 + \frac{5z^3}{16} - \frac{25z^9}{648} + \frac{47z^{11}}{1296} + \frac{z^{13}}{648} + \frac{35z}{24}$$
$$- \frac{z^{15}}{3888} - \frac{65z^5}{72} - \frac{35z^7}{48}\right)\alpha_3^4 + \left(-\frac{5}{9}z^7 - \frac{23}{648}z^9 + \frac{23}{9}z^3 + \frac{2}{405}z^{11} + \frac{31}{8}z\right.$$
$$\left.+ \frac{1}{3240}z^{13} - \frac{61}{72}z^5\right)\alpha_5\alpha_3^3 + \left(\left(\frac{7}{864}z^{11} - \frac{125}{144}z^7 - \frac{1}{5184}z^{13} + \frac{1445}{192}z - \frac{265}{192}z^5\right.\right.$$
$$\left.+ \frac{475}{96}z^3 - \frac{25}{1728}z^9\right)\alpha_4^2 + \left(\frac{31}{24}z^7 + \frac{1}{432}z^{13} - \frac{7}{144}z^{11} + \frac{77}{48}z^5 - \frac{1}{36}z^9 - \frac{53}{48}z^3\right.$$

$$-\frac{29}{8}z\Big)\alpha_4 + \left(\frac{7}{36}z^5 + \frac{1}{6}z^7 - \frac{1}{810}z^{11} - \frac{10}{9}z^3 + \frac{5}{648}z^9 - \frac{37}{24}z\right)\alpha_6 + \frac{z}{6} - \frac{z^9}{18}$$

$$-\frac{35z^5}{144} - \frac{5z^7}{12} - \frac{z^3}{36} + \frac{z^{11}}{12} - \frac{z^{13}}{144}\Bigg)\alpha_3^2 + \Bigg(\left(-\frac{45}{16}z^3 + \frac{7}{12}z^5 - \frac{1}{720}z^{11} - 4z\right.$$

$$\left.+ \frac{1}{108}z^9 + \frac{3}{8}z^7\right)\alpha_5\alpha_4 + \left(\frac{1}{120}z^9 + \frac{1}{120}z^{11} - \frac{17}{40}z^7 - \frac{21}{40}z^5 + \frac{1}{2}z^3 + \frac{3}{2}z\right)\alpha_5$$

$$+ \left(\frac{1}{3}z^3 + \frac{5}{12}z - \frac{11}{315}z^7 - \frac{1}{30}z^5 - \frac{1}{756}z^9\right)\alpha_7\Bigg)\alpha_3 + \Bigg(-\frac{55}{10368}z^9 - \frac{425}{384}z$$

$$-\frac{835}{1152}z^3 + \frac{35}{576}z^5 + \frac{35}{576}z^7 + \frac{1}{10368}z^{11}\Bigg)\alpha_4^3 + \Bigg(-\frac{23}{96}z^5 + \frac{103}{192}z^3 + \frac{7}{144}z^9$$

$$-\frac{11}{48}z^7 - \frac{1}{576}z^{11} + \frac{37}{32}z\Bigg)\alpha_4^2 + \Bigg(\left(-\frac{7}{180}z^7 + \frac{1}{540}z^9 + \frac{5}{6}z + \frac{7}{12}z^3 - \frac{1}{60}z^5\right)\alpha_6$$

$$+ \frac{5z^5}{32} + \frac{29z^7}{96} - \frac{13z^9}{96} - \frac{z}{4} - \frac{z^3}{24} + \frac{z^{11}}{96}\Bigg)\alpha_4 + \Bigg(-\frac{47}{1200}z^5 - \frac{13}{600}z^7 - \frac{1}{800}z^9$$

$$+ \frac{29}{120}z^3 + \frac{51}{160}z\Bigg)\alpha_5^2 + \left(\frac{1}{12}z^7 - \frac{1}{2}z - \frac{1}{90}z^9 - \frac{1}{4}z^3 + \frac{1}{15}z^5\right)\alpha_6 + \Bigg(-\frac{1}{480}z^5$$

$$-\frac{3}{32}z - \frac{7}{96}z^3 + \frac{11}{3360}z^7\Bigg)\alpha_8 + \frac{5z^9}{24} - \frac{5z^7}{16} - \frac{z^{11}}{48}$$

$$PP_7 := \Bigg(-\frac{1615}{1959552}z^{14} + \frac{4199}{559872}z^{12} + \frac{46189}{248832}z^8 - \frac{230945}{497664} + \frac{1}{11022480}z^{20}$$

$$-\frac{323}{4898880}z^{16} - \frac{2540395}{497664}z^2 + \frac{46189}{373248}z^{10} - \frac{1154725}{165888}z^4 - \frac{1062347}{497664}z^6$$

$$+ \frac{19}{22044960}z^{18}\Bigg)\alpha_3^7 + \Bigg(\Bigg(-\frac{17017}{27648}z^8 + \frac{17}{233280}z^{16} + \frac{85085}{55296} + \frac{85085}{4608}z^2$$

$$+ \frac{17}{7776}z^{14} + \frac{1446445}{55296}z^4 - \frac{221}{15552}z^{12} - \frac{1}{349920}z^{18} - \frac{46189}{124416}z^{10} + \frac{221221}{27648}z^6\Bigg)$$

$$\alpha_4 - \frac{305305z^4}{55296} - \frac{35035z^2}{6912} - \frac{35035}{55296} - \frac{z^{16}}{7776} + \frac{z^{18}}{58320} + \frac{19019z^8}{41472} + \frac{18733z^{10}}{62208}$$

$$+ \frac{767z^{12}}{93312} - \frac{241z^{14}}{46656} - \frac{155155z^6}{82944}\Bigg)\alpha_3^5 + \Bigg(\frac{13}{1944}z^{12} + \frac{2431}{13824}z^8 - \frac{25025}{6912}z^6$$

$$-\frac{35035}{3072}z^4 - \frac{35035}{4608}z^2 - \frac{1}{38880}z^{16} + \frac{715}{5184}z^{10} - \frac{5005}{9216} - \frac{1}{1944}z^{14}\Bigg)\alpha_5\alpha_3^4 + \Bigg(\Bigg($$

$$-\frac{25025}{18432} - \frac{35}{31104}z^{14} + \frac{1}{46656}z^{16} + \frac{65}{62208}z^{12} + \frac{7865}{13824}z^8 + \frac{33605}{124416}z^{10}$$

$$-\frac{725725}{27648}z^4 - \frac{325325}{18432}z^2 - \frac{55055}{6912}z^6\Bigg)\alpha_4^2 + \Bigg(\frac{19}{2592}z^{14} + \frac{13475}{1024}z^2 - \frac{1}{3888}z^{16}$$

$$+ \frac{66605}{13824}z^6 - \frac{1573}{2592}z^{10} + \frac{13475}{9216} + \frac{133}{7776}z^{12} - \frac{2695}{2304}z^8 + \frac{11165}{768}z^4\Bigg)\alpha_4$$

$$+ \Bigg(\frac{17017}{6912}z^2 + \frac{7007}{1728}z^4 + \frac{1001}{6912} - \frac{1001}{25920}z^8 + \frac{1}{7290}z^{14} + \frac{71071}{51840}z^6 - \frac{1573}{38880}z^{10}$$

$$-\frac{13}{11664}z^{12}\Bigg)\alpha_6 + \frac{4165z^8}{20736} + \frac{401z^{10}}{1152} - \frac{67375z^2}{55296} - \frac{21875z^6}{41472} - \frac{32375z^4}{27648}$$

$$-\frac{11z^{14}}{864} - \frac{13475}{55296} + \frac{z^{16}}{1296} - \frac{13z^{12}}{1728}\Bigg)\alpha_3^3 + \Bigg(\Bigg(-\frac{13}{6480}z^{12} + \frac{5005}{512}z^2 - \frac{143}{640}z^8$$

$$-\frac{143}{1152}z^{10} + \frac{1001}{64}z^4 + \frac{7007}{1440}z^6 + \frac{1001}{1536} + \frac{1}{4320}z^{14}\Big)\alpha_5\,\alpha_4 + \Big(-\frac{1}{360}z^{12}$$

$$-\frac{1561}{768}z^6 + \frac{283}{1440}z^{10} - \frac{245}{512} - \frac{1}{720}z^{14} - \frac{2695}{512}z^2 - \frac{1519}{256}z^4 + \frac{371}{960}z^8\Big)\alpha_5$$

$$+ \Big(\frac{1}{4536}z^{12} - \frac{11}{384} - \frac{55}{48}z^4 + \frac{11}{3360}z^8 - \frac{1177}{2880}z^6 + \frac{121}{15120}z^{10} - \frac{77}{128}z^2\Big)\alpha_7\Big)\alpha_3^2$$

$$+ \Big(\Big(\frac{25025}{13824}z^6 + \frac{143}{62208}z^{12} - \frac{1859}{13824}z^8 + \frac{35035}{9216}z^2 + \frac{5005}{18432} - \frac{1}{31104}z^{14}$$

$$+ \frac{55055}{9216}z^4 - \frac{715}{20736}z^{10}\Big)\alpha_4^3 + \Big(-\frac{1225}{2048} - \frac{6125}{1024}z^2 - \frac{35}{16}z^6 + \frac{1}{1728}z^{14}$$

$$-\frac{83}{3456}z^{12} - \frac{5285}{768}z^4 + \frac{889}{1536}z^8 + \frac{1225}{6912}z^{10}\Big)\alpha_4^2 + \Big(\Big(-\frac{231}{128}z^2 - \frac{77}{24}z^4 - \frac{77}{768}$$

$$-\frac{1}{1620}z^{12} + \frac{11}{320}z^8 - \frac{77}{72}z^6 + \frac{77}{4320}z^{10}\Big)\alpha_6 + \frac{1925}{6144} + \frac{47z^{12}}{576} - \frac{z^{14}}{288} - \frac{3z^{10}}{8}$$

$$-\frac{65z^8}{192} + \frac{5225z^4}{3072} + \frac{1925z^2}{1024} + \frac{3155z^6}{4608}\Big)\alpha_4 + \Big(-\frac{539}{960}z^6 - \frac{231}{128}z^4 - \frac{1309}{1280}z^2$$

$$+ \frac{143}{14400}z^{10} + \frac{1}{2400}z^{12} - \frac{77}{1280} + \frac{11}{960}z^8\Big)\alpha_5^2 + \Big(\frac{2093}{1152}z^4 - \frac{91}{864}z^8 + \frac{1127}{1728}z^6$$

$$+ \frac{1}{270}z^{12} - \frac{11}{240}z^{10} + \frac{1715}{1152}z^2 + \frac{245}{2304}\Big)\alpha_6 + \Big(\frac{13}{128}z^2 + \frac{1}{256} + \frac{1}{960}z^8 + \frac{15}{64}z^4$$

$$+ \frac{43}{480}z^6 - \frac{11}{10080}z^{10}\Big)\alpha_8 + \frac{85z^{10}}{192} - \frac{35z^2}{1024} - \frac{47z^6}{768} - \frac{17z^8}{128} - \frac{25z^4}{512} - \frac{35z^{12}}{288}$$

$$-\frac{35}{2048} + \frac{z^{14}}{144}\Big)\alpha_3 + \Big(\frac{143}{3840}z^8 - \frac{77}{64}z^2 - \frac{1}{5760}z^{12} - \frac{77}{120}z^6 - \frac{3157}{1536}z^4 - \frac{77}{1024}$$

$$+ \frac{11}{2160}z^{10}\Big)\alpha_5\,\alpha_4^2 + \Big(\Big(\frac{427}{576}z^6 + \frac{1}{480}z^{12} + \frac{245}{128}z^2 + \frac{7}{3}z^4 + \frac{245}{1536} - \frac{343}{1920}z^8$$

$$-\frac{19}{720}z^{10}\Big)\alpha_5 + \Big(-\frac{13}{3360}z^8 + \frac{17}{48}z^4 + \frac{1}{128} - \frac{1}{3024}z^{10} + \frac{11}{64}z^2 + \frac{31}{240}z^6\Big)\alpha_7\Big)\alpha_4$$

$$+ \Big(\Big(\frac{7}{16}z^4 - \frac{1}{480}z^8 + \frac{7}{48}z^6 + \frac{7}{640} - \frac{1}{900}z^{10} + \frac{7}{32}z^2\Big)\alpha_6 - \frac{3z^6}{16} - \frac{z^{12}}{160}$$

$$+ \frac{3z^8}{32} - \frac{77z^2}{128} + \frac{z^{10}}{20} - \frac{77}{1024} - \frac{257z^4}{512}\Big)\alpha_5 + \Big(\frac{1}{48}z^8 - \frac{35}{288}z^6 + \frac{1}{504}z^{10}$$

$$-\frac{5}{384} - \frac{65}{192}z^4 - \frac{15}{64}z^2\Big)\alpha_7 + \Big(-\frac{1}{108}z^2 - \frac{1}{3456} - \frac{23}{864}z^4 - \frac{19}{1620}z^6$$

$$-\frac{1}{12960}z^8\Big)\alpha_9$$

$$PP_8 := \Big(\frac{55}{839808}z^{17} + \frac{136675}{31104}z^7 - \frac{11}{264539520}z^{21} + \frac{11}{2519424}z^{19} - \frac{1510355}{93312}z^3$$

$$-\frac{2695}{139968}z^{13} - \frac{1925}{139968}z^{11} - \frac{1}{264539520}z^{23} + \frac{140525}{139968}z^9 - \frac{535535}{31104}z$$

$$-\frac{407}{419904}z^{15} + \frac{105875}{93312}z^5\Big)\alpha_3^8 + \Big(\Big(\frac{606725}{7776}z^3 + \frac{423395}{5184}z + \frac{1435}{23328}z^{13}$$

$$-\frac{11375}{648}z^7 - \frac{89425}{23328}z^9 + \frac{1}{6298560}z^{21} - \frac{1}{5184}z^{17} - \frac{1}{209952}z^{19} + \frac{31}{17496}z^{15}$$

$$-\frac{175}{192}z^5 + \frac{35}{432}z^{11}\Big)\alpha_4 - \frac{24395\,z^3}{2592} - \frac{97\,z^{15}}{87480} + \frac{10675\,z^5}{2592} - \frac{z^{21}}{1049760} + \frac{23\,z^{17}}{43740}$$

$$-\frac{3115\,z}{216} - \frac{791\,z^{13}}{11664} - \frac{1505\,z^{11}}{11664} + \frac{z^{19}}{116640} + \frac{3745\,z^7}{648} + \frac{9205\,z^9}{5832}\Big)\alpha_3^6 + \Big(\frac{175}{24}z^7$$

$$-\frac{41}{48600}z^{15} + \frac{539}{324}z^9 - \frac{7}{324}z^{13} - \frac{10759}{288}z^3 - \frac{49}{36}z^5 + \frac{1}{24300}z^{17} - \frac{679}{18}z$$

$$-\frac{49}{6480}z^{11} + \frac{1}{583200}z^{19}\Big)\alpha_5\,\alpha_3^5 + \Big(\Big(-\frac{1477}{10368}z^{11} + \frac{42875}{10368}z^9 + \frac{7}{62208}z^{17}$$

$$+\frac{35245}{1728}z^7 - \frac{791}{15552}z^{13} - \frac{2905}{576}z^5 - \frac{818825}{6912}z - \frac{1}{559872}z^{19} - \frac{260995}{2304}z^3$$

$$+\frac{1}{15552}z^{15}\Big)\alpha_4^2 + \Big(-\frac{245}{24}z^5 + \frac{287}{1944}z^{13} + \frac{1}{46656}z^{19} + \frac{2485}{48}z - \frac{2765}{648}z^9$$

$$+\frac{413}{864}z^{11} - \frac{1}{1296}z^{17} + \frac{20825}{576}z^3 - \frac{5}{1296}z^{15} - \frac{7315}{432}z^7\Big)\alpha_4 + \Big(\frac{91}{14580}z^{13}$$

$$+\frac{553}{324}z^5 - \frac{791}{324}z^7 + \frac{5159}{324}z^3 - \frac{847}{1458}z^9 + \frac{49}{14580}z^{11} - \frac{1}{87480}z^{17} + \frac{3409}{216}z$$

$$+\frac{1}{8748}z^{15}\Big)\alpha_6 - \frac{245\,z}{72} + \frac{17135\,z^9}{15552} + \frac{3565\,z^5}{1728} + \frac{4175\,z^7}{1728} - \frac{385\,z^3}{432} + \frac{23\,z^{15}}{5184}$$

$$-\frac{z^{19}}{15552} - \frac{193\,z^{13}}{1728} - \frac{2351\,z^{11}}{15552} + \frac{7\,z^{17}}{5184}\Big)\alpha_3^4 + \Big(\Big(\frac{1}{3645}z^{15} + \frac{497}{72}z^5 + \frac{77}{3240}z^{13}$$

$$-\frac{112}{9}z^7 - \frac{1}{38880}z^{17} - \frac{10087}{3888}z^9 + \frac{7}{162}z^{11} + \frac{2625}{32}z + \frac{4361}{54}z^3\Big)\alpha_5\,\alpha_4 + \Big($$

$$-\frac{319}{2160}z^{11} + \frac{323}{48}z^7 - \frac{829}{48}z^3 + \frac{51}{16}z^5 + \frac{1}{2160}z^{15} - \frac{93}{4}z - \frac{101}{2160}z^{13} + \frac{1}{6480}z^{17}$$

$$+\frac{749}{432}z^9\Big)\alpha_5 + \Big(\frac{323}{1944}z^9 + \frac{1}{1080}z^{11} + \frac{145}{216}z^7 - \frac{1}{40824}z^{15} - \frac{7}{8}z^5 - \frac{1169}{216}z^3$$

$$-\frac{373}{72}z - \frac{11}{9720}z^{13}\Big)\alpha_7\Big)\alpha_3^3 + \Big(\Big(-\frac{5915}{864}z^7 - \frac{11}{23328}z^{15} + \frac{127225}{2304}z$$

$$+\frac{1}{186624}z^{17} + \frac{581}{7776}z^{11} + \frac{44135}{864}z^3 + \frac{9065}{1728}z^5 + \frac{133}{15552}z^{13} - \frac{37835}{31104}z^9\Big)\alpha_4^3$$

$$+\Big(\frac{1425}{128}z^7 + \frac{55}{10368}z^{15} + \frac{1775}{384}z^5 - \frac{12845}{384}z^3 - \frac{1}{10368}z^{17} - \frac{1445}{32}z - \frac{481}{1152}z^{11}$$

$$+\frac{935}{384}z^9 - \frac{175}{3456}z^{13}\Big)\alpha_4^2 + \Big(\Big(-\frac{49}{12960}z^{13} - \frac{7}{432}z^{11} - \frac{3493}{144}z^3 + \frac{1}{9720}z^{15}$$

$$-\frac{119}{32}z^5 + \frac{1505}{2592}z^9 + \frac{49}{18}z^7 - \frac{805}{32}z\Big)\alpha_6 + \frac{11\,z^{11}}{24} - \frac{677\,z^5}{192} + \frac{71\,z^{13}}{576} - \frac{1255\,z^7}{288}$$

$$+\frac{203\,z}{24} - \frac{17\,z^{15}}{864} + \frac{z^{17}}{1728} + \frac{143\,z^3}{48} - \frac{3121\,z^9}{1728}\Big)\alpha_4 + \Big(-\frac{3843}{320}z - \frac{11669}{960}z^3$$

$$-\frac{1}{14400}z^{15} - \frac{91}{43200}z^{13} + \frac{581}{1728}z^9 + \frac{49}{43200}z^{11} - \frac{1393}{960}z^5 + \frac{1519}{960}z^7\Big)\alpha_5^2 + \Big($$

$$-\frac{13}{48}z^5 + \frac{121}{2160}z^{11} - \frac{1}{1620}z^{15} - \frac{115}{216}z^9 - \frac{49}{24}z^7 + \frac{23}{2160}z^{13} + \frac{37}{4}z + \frac{119}{16}z^3\Big)\alpha_6$$

$$+\Big(\frac{67}{192}z^5 - \frac{15}{112}z^7 - \frac{443}{12096}z^9 + \frac{151}{96}z^3 + \frac{11}{60480}z^{13} + \frac{99}{64}z - \frac{1}{4320}z^{11}\Big)\alpha_8$$

$$-\frac{z}{6}+\frac{131\,z^9}{288}+\frac{35\,z^5}{144}+\frac{35\,z^7}{96}+\frac{z^3}{36}+\frac{25\,z^{15}}{864}-\frac{23\,z^{13}}{144}-\frac{z^{11}}{72}-\frac{z^{17}}{864}\Bigg)\alpha_3^2$$

$$+\Bigg(\Bigg(\Bigg(\frac{1}{17280}\,z^{15}-\frac{427}{17280}\,z^{11}-\frac{413}{96}\,z^5-\frac{11921}{384}\,z^3-\frac{6349}{192}\,z-\frac{7}{3240}\,z^{13}+\frac{1043}{1728}\,z^9$$

$$+\frac{4025}{1152}\,z^7\Bigg)\alpha_5\,\alpha_4^2+\Bigg(\Bigg(\frac{151}{8}\,z^3-\frac{41}{32}\,z^5+\frac{37}{240}\,z^{11}-\frac{1}{1440}\,z^{15}-\frac{491}{96}\,z^7+\frac{53}{4320}\,z^{13}$$

$$+24\,z-\frac{155}{144}\,z^9\Bigg)\alpha_5+\Bigg(\frac{79}{16}\,z+\frac{1}{9072}\,z^{13}-\frac{263}{3024}\,z^9-\frac{107}{252}\,z^7+\frac{13}{7560}\,z^{11}$$

$$+\frac{115}{24}\,z^3+\frac{47}{48}\,z^5\Bigg)\alpha_7\Bigg)\alpha_4+\Bigg(\Bigg(-\frac{31}{60}\,z^7+\frac{1}{2700}\,z^{13}+\frac{673}{120}\,z^3+\frac{23}{4}\,z+\frac{1}{1080}\,z^{11}$$

$$+\frac{61}{60}\,z^5-\frac{59}{540}\,z^9\Bigg)\alpha_6+\frac{269\,z^5}{240}+\frac{701\,z^7}{480}+\frac{z^{15}}{480}+\frac{3\,z^9}{5}-\frac{4\,z^3}{3}-\frac{z^{13}}{48}-\frac{11\,z^{11}}{80}$$

$$-\frac{7\,z}{2}\Bigg)\alpha_5+\Bigg(\frac{193}{420}\,z^7-\frac{29}{2520}\,z^{11}+\frac{143}{1260}\,z^9-\frac{53}{24}\,z^3-\frac{5}{2}\,z-\frac{1}{1512}\,z^{13}$$

$$-\frac{11}{120}\,z^5\Bigg)\alpha_7+\Bigg(-\frac{23}{72}\,z-\frac{275}{864}\,z^3-\frac{19}{216}\,z^5+\frac{19}{3402}\,z^9+\frac{1}{38880}\,z^{11}$$

$$+\frac{59}{3024}\,z^7\Bigg)\alpha_9\Bigg)\alpha_3+\Bigg(-\frac{72695}{18432}\,z^3+\frac{35}{165888}\,z^{13}-\frac{1}{497664}\,z^{15}+\frac{6545}{165888}\,z^9$$

$$-\frac{30415}{6144}\,z+\frac{5215}{18432}\,z^7-\frac{343}{55296}\,z^{11}-\frac{4165}{6144}\,z^5\Bigg)\alpha_4^4+\Bigg(\frac{1255}{256}\,z^3+\frac{1}{20736}\,z^{15}$$

$$-\frac{225}{256}\,z^7-\frac{65}{432}\,z^9+\frac{371}{6912}\,z^{11}+\frac{65}{384}\,z^5-\frac{11}{3456}\,z^{13}+\frac{425}{64}\,z\Bigg)\alpha_4^3+\Bigg(\Bigg(\frac{829}{192}\,z^3$$

$$+\frac{37}{8640}\,z^{11}-\frac{71}{288}\,z^7-\frac{73}{1728}\,z^9+\frac{341}{64}\,z-\frac{1}{12960}\,z^{13}+\frac{43}{48}\,z^5\Bigg)\alpha_6-\frac{21\,z^{11}}{128}$$

$$-\frac{265\,z^3}{192}-\frac{z^{15}}{2304}+\frac{103\,z^9}{384}+\frac{1771\,z^7}{2304}-\frac{259\,z}{96}+\frac{329\,z^5}{768}+\frac{13\,z^{13}}{768}\Bigg)\alpha_4^2+\Bigg(\Bigg($$

$$-\frac{301}{5760}\,z^9+\frac{2429}{640}\,z+\frac{1151}{320}\,z^3+\frac{1}{9600}\,z^{13}+\frac{1}{4800}\,z^{11}+\frac{393}{640}\,z^5-\frac{163}{480}\,z^7\Bigg)\alpha_5^2+\Bigg($$

$$-\frac{43}{120}\,z^5-\frac{1}{36}\,z^{11}+\frac{17}{30}\,z^7+\frac{1}{1080}\,z^{13}-\frac{47}{12}\,z^3-5\,z+\frac{7}{60}\,z^9\Bigg)\alpha_6+\Bigg(-\frac{11}{64}\,z^5$$

$$-\frac{85}{128}\,z^3-\frac{11}{40320}\,z^{11}+\frac{139}{24192}\,z^9-\frac{311}{384}\,z+\frac{107}{4032}\,z^7\Bigg)\alpha_8+\frac{z}{4}-\frac{9\,z^9}{32}-\frac{5\,z^5}{32}$$

$$-\frac{43\,z^7}{192}+\frac{z^3}{24}+\frac{z^{15}}{576}+\frac{z^{11}}{4}-\frac{z^{13}}{24}\Bigg)\alpha_4+\Bigg(\frac{51}{1600}\,z^5-\frac{1}{1600}\,z^{13}-\frac{1}{192}\,z^{11}$$

$$-\frac{153}{80}\,z+\frac{51}{800}\,z^9-\frac{103}{64}\,z^3+\frac{279}{800}\,z^7\Bigg)\alpha_5^2+\Bigg(\frac{1}{5040}\,z^{11}-\frac{31}{48}\,z^3+\frac{31}{3780}\,z^9-\frac{2}{3}\,z$$

$$-\frac{3}{20}\,z^5+\frac{121}{2520}\,z^7\Bigg)\alpha_7\,\alpha_5+\Bigg(-\frac{11}{24}\,z^3+\frac{7}{1620}\,z^9-\frac{1}{4050}\,z^{11}+\frac{7}{360}\,z^7-\frac{67}{120}\,z$$

$$-\frac{13}{120}\,z^5\Bigg)\alpha_6^2+\Bigg(-\frac{13}{48}\,z^7+\frac{23}{36}\,z^3+\frac{11}{240}\,z^{11}-\frac{1}{360}\,z^{13}-\frac{67}{720}\,z^5-\frac{9}{80}\,z^9$$

$$+\frac{7}{6}\,z\Bigg)\alpha_6+\Bigg(-\frac{1}{80}\,z^9+\frac{31}{64}\,z^3+\frac{3}{40}\,z^5+\frac{11}{6720}\,z^{11}+\frac{9}{16}\,z-\frac{57}{1120}\,z^7\Bigg)\alpha_8$$

$$+\Bigg(\frac{3}{200}\,z^5+\frac{7}{120}\,z+\frac{17}{360}\,z^3-\frac{13}{12600}\,z^7-\frac{19}{56700}\,z^9\Bigg)\alpha_{10}+\frac{35\,z^9}{128}-\frac{z^{15}}{384}$$

$$-\frac{35\,z^{11}}{128} + \frac{7\,z^{13}}{128}$$

Comarison with Hall's method:

Hall_PP[1],...,Hall_PP[5] obtained with Hall's method

**Due to computer resources, we computed only
the first 5 polynomials with Hall's method**

First 5 polynomials obtained with Hall's method:

```
> Hall_PP[1]:=(1/6+1/3*z^2)*alpha[3];
  print();

  Hall_PP[2]:=(1/6*z-1/9*z^3-1/18*z^5)*alpha[3]^2+(-1/4*z+1/12*
  z^3)*alpha[4]-1/2*z^3;
  print();

  Hall_PP[3]:=collect((-35/432-35/216*z^4-175/432*z^2+7/324*
  z^6+1/162*z^8)*alpha[3]^3+((5/48+5/8*z^2+5/24*z^4-1/36*z^6)*
  alpha[4]-1/16-1/8*z^2-1/4*z^4+1/6*z^6)*alpha[3]+(-1/20*z^4-1/40
  -1/5*z^2)*alpha[5],{alpha[3],alpha[4],alpha[5]});
  print();

  Hall_PP[4]:=(-35/72*z+25/108*z^5-5/216*z^3-1/1944*z^11+5/108*z^7
  -5/1944*z^9)*alpha[3]^4+((1/216*z^9-5/12*z^5-1/18*z^7+1/6*
  z^3+29/24*z)*alpha[4]-1/6*z+1/36*z^3-1/36*z^9+11/36*z^5+1/12*
  z^7)*alpha[3]^2+(-1/12*z^3-1/2*z+2/15*z^5+1/60*z^7)*alpha[5]*
  alpha[3]+(-37/96*z-11/96*z^3+7/96*z^5-1/288*z^7)*alpha[4]^2+
  (-1/4*z^5+1/24*z^3+1/24*z^7+1/4*z)*alpha[4]+(1/18*z^3+1/6*
  z-1/45*z^5)*alpha[6]-1/8*z^7+3/8*z^5;
  print();

  Hall_PP[5]:=(7007/6912*z^4+1001/864*z^2+1001/10368*z^6-143/2592*
  z^8+13/58320*z^12+1001/6912-143/19440*z^10+1/29160*z^14)*alpha
  [3]^5+((11/1296*z^10-385/128*z^2-385/144*z^4-385/1728*z^6+11/96*
  z^8-1/1944*z^12-385/1152)*alpha[4]-1/72*z^10-17/108*z^8+385/576*
  z^4+175/1152+875/1152*z^2+35/432*z^6+1/324*z^12)*alpha[3]^3+
  (-1/30*z^8+7/64+49/480*z^6+77/64*z^2-1/360*z^10+35/32*z^4)*alpha
  [5]*alpha[3]^2+((35/256+1/864*z^10+175/128*z^2+245/192*z^4
```

```
-7/192*z^8+7/96*z^6)*alpha[4]^2+(-25/128-1/72*z^10-5/96*z^6
-75/64*z^2+3/16*z^8-185/192*z^4)*alpha[4]+(-49/144*z^2-7/288
-7/216*z^6-49/144*z^4+1/135*z^8)*alpha[6]+1/24*
z^10+25/768+25/384*z^2+3/16*z^6+3/32*z^4-5/16*z^8)*alpha[3]+
(-7/16*z^4-7/192-7/360*z^6+1/240*z^8-7/16*z^2)*alpha[5]*alpha[4]
+(9/32*z^4+3/64-1/40*z^8+3/8*z^2)*alpha[5]+(11/168*z^4+3/56*
z^2+1/336+1/252*z^6)*alpha[7];
```

$$Hall\_PP_1 := \left(\frac{1}{6} + \frac{z^2}{3} \right) \alpha_3 \tag{17}$$

$$Hall\_PP_2 := \left(-\frac{1}{9} z^3 - \frac{1}{18} z^5 + \frac{1}{6} z \right) \alpha_3^2 + \left(-\frac{1}{4} z + \frac{1}{12} z^3 \right) \alpha_4 - \frac{z^3}{2}$$

$$Hall\_PP_3 := \left(\frac{1}{162} z^8 - \frac{35}{432} - \frac{35}{216} z^4 - \frac{175}{432} z^2 + \frac{7}{324} z^6 \right) \alpha_3^3 + \left(\left(\frac{5}{8} z^2 - \frac{1}{36} z^6 \right. \right.$$
$$\left. + \frac{5}{24} z^4 + \frac{5}{48} \right) \alpha_4 - \frac{1}{16} - \frac{z^4}{4} + \frac{z^6}{6} - \frac{z^2}{8} \right) \alpha_3 + \left(-\frac{1}{40} - \frac{1}{20} z^4 - \frac{1}{5} z^2 \right) \alpha_5$$

$$Hall\_PP_4 := \left(\frac{25}{108} z^5 + \frac{5}{108} z^7 - \frac{5}{1944} z^9 - \frac{1}{1944} z^{11} - \frac{35}{72} z - \frac{5}{216} z^3 \right) \alpha_3^4 + \left(\left(-\frac{1}{18} z^7 \right. \right.$$
$$\left. + \frac{1}{6} z^3 + \frac{29}{24} z - \frac{5}{12} z^5 + \frac{1}{216} z^9 \right) \alpha_4 - \frac{z}{6} + \frac{z^7}{12} + \frac{z^3}{36} - \frac{z^9}{36} + \frac{11 z^5}{36} \right) \alpha_3^2 + \left(\right.$$
$$\left. -\frac{1}{2} z - \frac{1}{12} z^3 + \frac{2}{15} z^5 + \frac{1}{60} z^7 \right) \alpha_5 \alpha_3 + \left(\frac{7}{96} z^5 - \frac{1}{288} z^7 - \frac{37}{96} z - \frac{11}{96} z^3 \right) \alpha_4^2$$
$$+ \left(\frac{1}{4} z - \frac{1}{4} z^5 + \frac{1}{24} z^3 + \frac{1}{24} z^7 \right) \alpha_4 + \left(\frac{1}{18} z^3 - \frac{1}{45} z^5 + \frac{1}{6} z \right) \alpha_6 + \frac{3 z^5}{8} - \frac{z^7}{8}$$

$$Hall\_PP_5 := \left(\frac{13}{58320} z^{12} + \frac{1}{29160} z^{14} + \frac{7007}{6912} z^4 + \frac{1001}{10368} z^6 - \frac{143}{19440} z^{10} - \frac{143}{2592} z^8 \right.$$
$$\left. + \frac{1001}{6912} + \frac{1001}{864} z^2 \right) \alpha_3^5 + \left(\left(-\frac{385}{1728} z^6 - \frac{1}{1944} z^{12} - \frac{385}{128} z^2 + \frac{11}{1296} z^{10} - \frac{385}{144} z^4 \right. \right.$$
$$\left. - \frac{385}{1152} + \frac{11}{96} z^8 \right) \alpha_4 + \frac{385 z^4}{576} + \frac{35 z^6}{432} + \frac{175}{1152} - \frac{z^{10}}{72} + \frac{875 z^2}{1152} - \frac{17 z^8}{108} + \frac{z^{12}}{324} \right)$$
$$\alpha_3^3 + \left(\frac{49}{480} z^6 + \frac{77}{64} z^2 + \frac{35}{32} z^4 - \frac{1}{30} z^8 + \frac{7}{64} - \frac{1}{360} z^{10} \right) \alpha_5 \alpha_3^2 + \left(\left(\frac{35}{256} + \frac{7}{96} z^6 \right. \right.$$
$$\left. - \frac{7}{192} z^8 + \frac{175}{128} z^2 + \frac{1}{864} z^{10} + \frac{245}{192} z^4 \right) \alpha_4^2 + \left(\frac{3}{16} z^8 - \frac{185}{192} z^4 - \frac{75}{64} z^2 - \frac{1}{72} z^{10} \right.$$
$$\left. - \frac{5}{96} z^6 - \frac{25}{128} \right) \alpha_4 + \left(-\frac{49}{144} z^2 + \frac{1}{135} z^8 - \frac{49}{144} z^4 - \frac{7}{216} z^6 - \frac{7}{288} \right) \alpha_6 + \frac{25 z^2}{384}$$
$$+ \frac{25}{768} + \frac{z^{10}}{24} + \frac{3 z^6}{16} - \frac{5 z^8}{16} + \frac{3 z^4}{32} \right) \alpha_3 + \left(-\frac{7}{192} - \frac{7}{16} z^4 - \frac{7}{16} z^2 + \frac{1}{240} z^8 \right.$$
$$\left. - \frac{7}{360} z^6 \right) \alpha_5 \alpha_4 + \left(\frac{3}{8} z^2 + \frac{3}{64} - \frac{1}{40} z^8 + \frac{9}{32} z^4 \right) \alpha_5 + \left(\frac{1}{252} z^6 + \frac{3}{56} z^2 + \frac{11}{168} z^4 \right.$$
$$\left. + \frac{1}{336} \right) \alpha_7$$

Comparison: Chung versus Hall

```
> i3:='i3':
  for i3 from 1 to min(number_polynomials,5) do:
```

```
    difference[i3]:=expand(Hall_PP[i3]-PP[i3]):
    print(i3,difference[i3]);
end:
```

$$1, 0$$ **(18)**
$$2, 0$$
$$3, 0$$
$$4, 0$$
$$5, 0$$