SUPPLEMENT A OF THE PAPER "TWO-SAMPLE KOLMOGOROV-SMIRNOV TYPE TESTS REVISITED: OLD AND NEW TESTS IN TERMS OF LOCAL LEVELS":

PROOFS AND COMPUTATION OF GLOBAL LEVELS*

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In this supplement we provide additional technical material to [7]. In Section A1 we prove Lemma 3.1. Section A2 focuses on the computation of global levels. Proofs of asymptotic results in Subsections 3.2 and 4.2 are given in Section A3. Section A4 provides technical results for proofs in Section A3.

A1. Proof of Lemma 3.1.

Proof of Lemma 3.1. Since weighted KS tests reject the corresponding null hypothesis if $KS^{w,\langle\rangle}_{m,n} > b$, the acceptance region of the form (2.2) is defined in terms of critical values

$$c_s = \frac{sm}{m+n} - \sqrt{\frac{mn}{m+n}}\,w\Big(\frac{s}{m+n}\Big)b, \quad s \in I_{m,n}, \tag{A1.1}$$

$$d_s = \frac{sm}{m+n} + \sqrt{\frac{mn}{m+n}}\,w\Big(\frac{s}{m+n}\Big)b, \quad s \in I_{m,n}, \tag{A1.2}$$

in the two-sided case. In the one-sided case d_s is replaced by $d_s = \min(s, m)$ for $s \in I_{m,n}$. The aforementioned critical values always fulfill $c_s < d_s$, $s \in I_{m,n}$. Typically, weighted KS critical values defined in (A1.1), (A1.2) are not proper but can easily be replaced by the proper critical values

$$\tilde{c}_s = \max\{0, s-n, \lceil c_s \rceil\} \ \text{ and } \ \tilde{d}_s = \min\{s, m, \lfloor d_s \rfloor\}, \quad s \in I_{m,n},$$

where $\lfloor x \rfloor$ ($\lceil x \rceil$) denotes the largest (smallest) integer less (greater) than or equal to x for $x \in \mathbb{R}$.

First, we focus on $\tilde{c}_s$, $s \in I_{m,n}$. Suppose there exists an $s \in I_{m,n}$ such that $c_s > \max(0, s-n)$, otherwise $\tilde{c}_s = \max(0, s-n)$, $s \in I_{m,n}$, which implies

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the assertion. For the sake of simplicity we consider c_s defined in (A1.1) as a continuous function for $s \in (0, m+n)$. Then we get $\lim_{s\to 0} c_s < 0$ and $\lim_{s\to m+n} c_s < m$. This together with the convexity of c_s implies that there exist unique $s_1, s_2 \in (0, m+n)$ such that $c_{s_1} = 0$ and $c_{s_2} = s_2 - n$. Moreover, it follows

(*a*) $\tilde{c}_s = 0$ for $s \in I_{m,n} \cap (0, s_1]$,
(*b*) $\tilde{c}_s = \lceil c_s \rceil$ for $s \in I_{m,n} \cap (s_1, s_2)$,
(*c*) $\tilde{c}_s = s - n$ for $s \in I_{m,n} \cap [s_2, m+n)$.

Obviously, (2.3) and (2.4) are fulfilled so that it suffices to check (2.5) for $\tilde{c}_s$, $s \in I_{m,n} \cap (s_1, s_2)$. Since c_s is convex, $c_s > s - n$ for $s \in (s_1, s_2)$ and $c_{s_2} = s_2 - n$, the slope of any secant of the curve c_s in (s_1, s_2), i.e., $(c_s - c_{s'})/(s - s')$, is less than the slope of the line $s - n$. Hence, $c_{s+1} - c_s \leq 1$ for any s such that $s, s+1 \in I_{m,n} \cap (s_1, s_2)$, i.e., (2.5) follows. Upper critical values can be treated similarly. $\square$

A2. Computation of global levels. Consider a two-sample GOF test with the acceptance region $A_{m,n}$ of the form (2.2) and critical values c_s, d_s, $s \in I_{m,n}$. It is not required that critical values c_s, d_s, $s \in I_{m,n}$, are proper. In order to compute the global level, i.e., $\mathbb{P}_0(V_m \notin A_{m,n}) = 1 - \mathbb{P}_0(V_m \in A_{m,n})$, we define

$$A_{m,n}^{(s)} = \{x \in \mathbb{N}_0^{m+n-1} : c_1 \leq x_1 \leq d_1, \ldots, c_s \leq x_s \leq d_s\}, \quad s \in I_{m,n}.$$

Thereby, $A_{m,n}^{(m+n-1)} \equiv A_{m,n}$. The acceptance probability $\mathbb{P}_0(V_m \in A_{m,n})$ can be computed in a sequential way. First, note that

$$\mathbb{P}_0(V_{m,1} = 0, V_m \in A_{m,n}^{(1)}) = \frac{n}{m+n} \mathbb{I}(c_1 \leq 0 \leq d_1)$$

and

$$\mathbb{P}_0(V_{m,1} = 1, V_m \in A_{m,n}^{(1)}) = \frac{m}{m+n} \mathbb{I}(c_1 \leq 1 \leq d_1),$$

where $\mathbb{I}(c_1 \leq j \leq d_1) = 1$ if $c_1 \leq j \leq d_1$ and $\mathbb{I}(c_1 \leq j \leq d_1) = 0$ else. Furthermore, for $c_{s+1} \leq j \leq d_{s+1}$ and $1 \leq s \leq m+n-2$ we get

$$\begin{aligned}
&\mathbb{P}_0(V_{m,s+1} = j, V_m \in A_{m,n}^{(s+1)}) \\
&\qquad = \mathbb{P}_0(V_{m,s+1} = j | V_{m,s} = j)\mathbb{P}_0(V_{m,s} = j, V_m \in A_{m,n}^{(s)}) \\
&\qquad\quad + \mathbb{P}_0(V_{m,s+1} = j | V_{m,s} = j-1)\mathbb{P}_0(V_{m,s} = j-1, V_m \in A_{m,n}^{(s)}) \\
&\qquad = \frac{n-s+j}{m+n-s}\mathbb{P}_0(V_{m,s} = j, V_m \in A_{m,n}^{(s)}) \\
&\qquad\quad + \frac{m-j+1}{m+n-s}\mathbb{P}_0(V_{m,s} = j-1, V_m \in A_{m,n}^{(s)}).
\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\mathbb{P}_0(V_m \in A_{m,n}) &= \mathbb{P}_0(V_{m,m+n-1} = m-1, V_m \in A_{m,n}^{(m+n-1)}) \\ &\quad + \mathbb{P}_0(V_{m,m+n-1} = m, V_m \in A_{m,n}^{(m+n-1)}).\end{aligned}$$

REMARK A2.1. *The acceptance probability $\mathbb{P}_0(V_m \in A_{m,n})$ can alternatively be calculated as a fraction of the number of V_m–paths, which lie in the acceptance region $A_{m,n}$, and the number of all possible V_m–paths. Noting that the number of different paths of the random process V_m is $\binom{m+n}{m}$, it suffices to count paths that belong to $A_{m,n}$. For example, see [11] for KS global levels.*

A3. Proofs of asymptotic results in Subsections 3.2 and 4.2. W.l.o.g. we assume that the underlying probability measure is given by

$$\mathbb{P}_0^* \text{ such that } F(t) = G(t) = t, \quad t \in [0,1]. \tag{A3.1}$$

The cases $\nu \in [0, 0.5)$ and $\nu = 0.5$. First, we define local weighted KS statistics as

$$KS^{\nu}_{m,n}(t) = D^{\nu}_{m,n}(t)(\sqrt{\zeta_{m,n}}G^{*,\nu}_n(t) - \sqrt{1-\zeta_{m,n}}F^{*,\nu}_m(t)), \tag{A3.2}$$

where $\zeta_{m,n} = m/(m+n)$,

$$G^{*,\nu}_n(t) = \sqrt{n}\frac{\hat{G}_n(t) - t}{(t(1-t))^{\nu}}, \quad F^{*,\nu}_m(t) = \sqrt{m}\frac{\hat{F}_m(t) - t}{(t(1-t))^{\nu}} \tag{A3.3}$$

and

$$D^{\nu}_{m,n}(t) = \Big(\frac{t(1-t)}{\hat{H}_{m+n}(t)(1-\hat{H}_{m+n}(t))}\Big)^{\nu}, \quad t \in [t_1, t_{m+n}). \tag{A3.4}$$

Then $KS^{\nu,\langle\rangle}_{m,n} = \sup_{t\in(0,1)}\langle KS^{\nu}_{m,n}(t)\rangle$ for $KS^{\nu,\langle\rangle}_{m,n}$ defined in (3.4). Obviously, for any interval $T \subset (0,1)$ we get

$$\begin{aligned}\sup_{t\in(0,1)\setminus T}|KS^{\nu}_{m,n}(t)| &\le \sup_{t\in[t_1,t_{m+n})} D^{\nu}_{m,n}(t)\Big(\sqrt{\zeta_{m,n}}\sup_{t\in(0,1)\setminus T}|G^{*,\nu}_n(t)| \\ &\quad + \sqrt{1-\zeta_{m,n}}\sup_{t\in(0,1)\setminus T}|F^{*,\nu}_m(t)|\Big).\end{aligned} \tag{A3.5}$$

Let $\mathbb{B}^{1,\nu}_n(t)$ and $\mathbb{B}^{2,\nu}_m(t)$ be weighted Brownian bridges as in Theorem A4.1. Then we get an alternative representation for local weighted KS statistics, i.e.,

$$\begin{aligned}KS^{\nu}_{m,n}(t) &= D^{\nu}_{m,n}(t)(\sqrt{\zeta_{m,n}}[G^{*,\nu}_n(t) - \mathbb{B}^{1,\nu}_n(t)] \\ &\quad - \sqrt{1-\zeta_{m,n}}[F^{*,\nu}_m(t) - \mathbb{B}^{2,\nu}_m(t)] + \mathbb{B}^{\nu}_{m,n}(t)),\end{aligned} \tag{A3.6}$$

where $\mathbb{B}^{\nu}_{m,n}(t) = \sqrt{\zeta_{m,n}}\mathbb{B}^{1,\nu}_{n}(t) - \sqrt{1-\zeta_{m,n}}\mathbb{B}^{2,\nu}_{m}(t)$, $m,n \in \mathbb{N}$, are weighted Brownian bridges.

Proof of Theorem 3.1. (a) First, we show

$$\sup_{t\in(0,1)\setminus T_{m,n}} |KS^{\nu}_{m,n}(t)| = o_{\mathbb{P}^*_0}(1), \tag{A3.7}$$

where

$$T_{m,n} = (\log(m+n)/(m+n), 1-\log(m+n)/(m+n)). \tag{A3.8}$$

Consider (A3.5) with $T = T_{m,n}$. Theorem 2.1 in [3] yields that

$$\sup_{t\in(0,1)\setminus T_{m,n}} |G^{*,\nu}_{n}(t)| = o_{\mathbb{P}^*_0}(1) \quad \text{and} \quad \sup_{t\in(0,1)\setminus T_{m,n}} |F^{*,\nu}_{m}(t)| = o_{\mathbb{P}^*_0}(1),$$

which together with Lemma A4.1 lead to (A3.7).

Next we consider (A3.6) for $t \in T_{m,n}$. By Lemma A4.2 the auxiliary variable $D^{\nu}_{m,n}(t)$ converges to one at least stochastically and uniformly in $t \in T_{m,n}$. Moreover, (a) in Theorem A4.1 yields that the first two terms in brackets in (A3.6) converge to zero stochastically and uniformly in $t \in (0,1)$. Therefore,

$$\sup_{t\in T_{m,n}} |KS^{\nu}_{m,n}(t) - \mathbb{B}^{\nu}_{m,n}(t)| = o_{\mathbb{P}^*_0}(1). \tag{A3.9}$$

Furthermore, by combining Theorem 2.1 in [3] and (a) in Theorem A4.1, we get

$$\sup_{t\in(0,1)\setminus T_{m,n}} |\mathbb{B}^{\nu}_{m,n}(t)| = o_{\mathbb{P}^*_0}(1). \tag{A3.10}$$

Finally, (A3.7), (A3.9) and (A3.10) lead to

$$\sup_{t\in(0,1)} |KS^{\nu}_{m,n}(t) - \mathbb{B}^{\nu}_{m,n}(t)| = o_{\mathbb{P}^*_0}(1). \tag{A3.11}$$

Consequently, for any $x \in \mathbb{R}$ we get

$$\lim_{m\to\infty} \mathbb{P}^*_0\Big(\sup_{t\in(0,1)} \langle KS^{\nu}_{m,n}(t)\rangle \le x\Big) = \lim_{m\to\infty} \mathbb{P}^*_0\Big(\sup_{t\in(0,1)} \langle \mathbb{B}^{\nu}_{m,n}(t)\rangle \le x\Big).$$

(b) For any $t \in (0,1)$ and $b \in \mathbb{R}$ we get by (A3.11) that

$$\mathbb{P}_0(KS^{\nu}_{m,n}(t) > b) = \mathbb{P}_0(\mathbb{B}^{\nu}_{m,n}(t) > b + o_{\mathbb{P}^*_0}(1))$$

and hence, $\mathbb{P}_0(KS^{\nu}_{m,n}(t) > b) \to \mathbb{P}_0(\mathbb{B}^{\nu}_{m,n}(t) > b)$ as $m \to \infty$. Noting that $\mathbb{P}_0(\mathbb{B}^{\nu}_{m,n}(t) > b) = 1 - \Phi(b/(t(1-t))^{1/2-\nu})$, we get the desired representation.
(c) Since $\sup_{t\in(0,1)} |\mathbb{B}^{\nu}_{m,n}(t)|$ is a non-degenerate random variable, e.g., cf. Theorem 4.2.3 in [2], the assertion immediately follows. □

PROOF OF THEOREM 3.2. (a) W.l.o.g. we assume that $n \equiv n(m) \geq m$, $m \in \mathbb{N}$. First, we show that

$$\lim_{m,n\to\infty} \mathbb{P}_0^*\Big(\sup_{t\in(0,1)\setminus T_m} |KS_{m,n}^{0.5}(t)| \leq b_m(x)\Big) = 1. \tag{A3.12}$$

We consider (A3.5) with $T = T_m$. Applying Lemma 4.4.4 in [2] to the second term in brackets and Lemma A4.1 to the term out of brackets in (A3.5), we get

$$\sup_{t\in(0,1)\setminus T_m} |KS_{m,n}^{0.5}(t)| \leq \\ O_{\mathbb{P}_0^*}(\sqrt{\zeta_{m,n}}) \sup_{t\in(0,1)\setminus T_m} |G_n^{*,0.5}(t)| + O_{\mathbb{P}_0^*}(\sqrt{\log_3(m)}). \tag{A3.13}$$

In order to prove (A3.12) it suffices to show that the right-hand side of (A3.13) is $o_{\mathbb{P}_0^*}(\sqrt{\log_2(m)})$. We consider two cases, (i) $m = o(n/\log_2(n))$ and (ii) $n/\log_2(n) = O(m)$.

(i) Applying Taylor series we get $\sqrt{\zeta_{m,n}} = o(1/\log_2(n))$. It can easily be seen, cf. e.g. the first theorem in [12], that $\sup_{t\in(0,1)} |G_n^{*,0.5}(t)| = \sqrt{2\log_2(n)}(1 + o_{\mathbb{P}_0^*}(1))$. Hence, the right-hand side in (A3.13) is equal to $O_{\mathbb{P}_0^*}(\sqrt{\log_3(m)})$, which implies (A3.12).

(ii) Lemma 4.4.4 in [2] yields $\sup_{t\in(0,1)\setminus T_m} |G_n^{*,0.5}(t)| = O_{\mathbb{P}_0^*}(\sqrt{\log_2(\delta_n)})$ for $\delta_n = (n/m)\log^5(m)$. Since $\delta_n \leq \log^6(n)$ for larger values of n, we get $\sup_{t\in(0,1)\setminus T_m} |G_n^{*,0.5}(t)| = O_{\mathbb{P}_0^*}(\sqrt{\log_3(n)})$. Consequently, the right-hand side in (A3.13) is $O_{\mathbb{P}_0^*}(\sqrt{\log_3(n)})$. Noting that $\sqrt{\log_3(m)} = \sqrt{\log_3(n)}(1+o(1))$, we get (A3.12).

Now we show that

$$\sup_{t\in T_m} |KS_{m,n}^{0.5}(t) - \mathbb{B}_{m,n}^{0.5}(t)| = o_{\mathbb{P}_0^*}(1/\sqrt{\log_2(m)}). \tag{A3.14}$$

We consider (A3.6) for $t \in T_m$. Since $T_m \subseteq T_n$, the first two terms in round brackets in (A3.6) are equal to $o(1/\sqrt{\log_2(m)})$ $\mathbb{P}_0^*$-almost surely and uniformly in $t \in T_m$, cf. (b) in Theorem A4.1. Together with Lemma A4.2 we get

$$KS_{m,n}^{0.5}(t) = [1 + O_{\mathbb{P}_0^*}(\sqrt{\log_2(m)/\log(m)})][o_{\mathbb{P}_0^*}(1/\sqrt{\log_2(m)}) + \mathbb{B}_{m,n}^{0.5}(t)]$$

uniformly in $t \in T_m$. Noting that $\sup_{t\in T_m} |\mathbb{B}_{m,n}^{0.5}(t)| = \sqrt{2\log_2(m)}(1 + o_{\mathbb{P}_0^*}(1))$, we arrive at (A3.14). Together with (A3.12) we obtain

$$\begin{aligned}\lim_{m\to\infty} \mathbb{P}_0^*(KS_{m,n}^{\langle\rangle,0.5} \leq b_m(x)) &= \lim_{m\to\infty} \mathbb{P}_0^*\Big(\sup_{t\in T_m} \langle KS_{m,n}^{0.5}(x)\rangle \leq b_m(t)\Big) \\ &= \lim_{m\to\infty} \mathbb{P}_0^*\Big(\sup_{t\in T_m} \langle \mathbb{B}^{0.5}(x)\rangle \leq b_m(t)\Big),\end{aligned}$$

where $\mathbb{B}^{0.5}$ is a normalized Brownian bridge. For the asymptotics of a normalized Brownian bridge, e.g., see (11)–(13) together with (15),(16) in [9].

(b) Due to (A3.14), we obtain $\mathbb{P}_0(KS^{0.5}_{m,n}(t) > b_m(x)) = 1 - \Phi(b_m(t) + o_{\mathbb{P}_0^*}(1))$ for $t \in T_m$ and $x \in \mathbb{R}$. Applying Mill's ratio, we get

$$\mathbb{P}_0(KS^{0.5}_{m,n}(t) > b_m(x)) = \frac{\exp(-x)}{2\log_2(m)\log(m)}\Big(1 + o\Big(\frac{\log_3(m)}{\log_2(m)}\Big)\Big).$$

Hence, for $s \in I_{m,n}$ with $\lim_{m,n\to\infty} \mathbb{P}_0(t_s \in T_m) = 1$, e.g., for $s > (m + n)\log^k(m)/m$ with $k \geq 6$, we get the assertions.

Now we focus on the left-tail case with $s \in I_{m,n}$ such that $\lim_{m\to\infty} s/((m+n)\log_2^3(m)/m) = \infty$ and $s \leq (m+n)\log^k(m)/m$ for an arbitrary but fixed $k \in \mathbb{N}$. The right-tail case can be proved similarly. Let c_s be defined in (A1.1) with $b \equiv b_m(x^{\langle\rangle}_\alpha)$ and let $F_{Bin}(\cdot|s, m/(m+n))$ denote the CDF of the binomial distribution with parameters s and $m/(m+n)$. With some analysis we obtain that

$$\alpha^{low}_s / F_{Bin}(c_s - 1|s, m/(m+n)) \to 1 \quad \text{as} \quad m \to \infty$$

uniformly on the range of s-values considered here. Similarly to the one-sample case, cf. Theorem 3.2 in [10], we obtain

$$\frac{F_{Bin}(c_s - 1|s, m/(m+n))}{1 - \Phi(b_m(x^{\langle\rangle}_\alpha)} \to 1 \quad \text{as} \quad m \to \infty$$

uniformly in s fulfilling (3.7). Following the proof of Lemma 4.3 in [10], we obtain the desired representation.

(c) The assertion follows by the considerations in (a), e.g., cf. (A3.14), and the knowledge about the sensitivity range of the one-sample HC test that can be find in [6] and [12]. □

The case $\nu \in (0.5, 1]$.

PROOF OF THEOREM 3.3. Let $n \equiv n(m)$, $m \in \mathbb{N}$. Similarly to (A3.2) we consider renormalized local statistics

$$\overline{KS}^{\nu}_{m,n}(t) = D^{\nu}_{m,n}(t)[(\zeta_{m,n})^{1-\nu} G^{\#,\nu}_n(t) - (1 - \zeta_{m,n})^{1-\nu} F^{\#,\nu}_m(t)] \tag{A3.15}$$

with one-sample renormalized weighted local KS statistics

$$G^{\#,\nu}_n(t) = n^{1-\nu}\frac{\hat{G}_n(t) - t}{(t(1-t))^{\nu}} \quad \text{and} \quad F^{\#,\nu}_m(t) = m^{1-\nu}\frac{\hat{F}_m(t) - t}{(t(1-t))^{\nu}}.$$

Representation (A3.15) together with Lemma A4.1 and results in [14] (see also [3]) imply that $\overline{KS}^{\nu,\langle\rangle}_{m,n} = \sup_{t\in(0,1)}\langle \overline{KS}^{\nu}_{m,n}(t)\rangle$ converges to a non-degenerate random variable. W.l.o.g. we assume $\lim_{m\to\infty}\zeta_{m,n} = p$. We distinguish two cases, (i) $p\in(0,1)$ and (ii) $p\in\{0,1\}$.
(i) With Lemma A4.1 and results in [14] we get

$$\sup_{t\in(k_m/m,1-k_m/m)} |\overline{KS}^{\nu}_{m,n}(t)| = o_{\mathbb{P}_0^*}(1)$$

for $1\le k_m\le m$, $k_m\to\infty$, $k_m/m\to 0$, $m\to\infty$. Asymptotically, extreme order statistics determine the supremum, i.e., for any $\epsilon>0$ there exists $s_0\equiv s_0(\epsilon)\in\mathbb{N}$ such that

$$\sup_{s\in I_{m,n}}\langle\overline{KS}^{\nu}_{m,n}(t_s)\rangle = \sup_{s\in[1,s_0]\cup[m+n-s_0,m+n]}\langle\overline{KS}^{\nu}_{m,n}(t_s)\rangle + O_{\mathbb{P}_0^*}(\epsilon) \tag{A3.16}$$

for sufficiently large m,n. In jump point t_s, $\overline{KS}^{\nu}_{m,n}$ can be rewritten as

$$\overline{KS}^{\nu}_{m,n}(t_s) = \frac{1}{(\zeta_{m,n}(1-\zeta_{m,n}))^{\nu}}\frac{s\zeta_{m,n}-V_{m,s}}{s^{\nu}}[1+O(s/n)].$$

Since $V_{m,s}$ converges in distribution to Y_s uniformly in $s\le s_0$, we get

$$\lim_{m\to\infty}\sup_{s\in[1,s_0]}\langle\overline{KS}^{\nu}_{m,n}(t_s)\rangle \overset{\mathcal{D}}{=} \frac{1}{(p(1-p))^{\nu}}\sup_{s\in[1,s_0]}\frac{\langle sp-Y_s\rangle}{s^{\nu}}.$$

Together with (A3.16) and the fact that $\overline{KS}^{\nu}_{m,n}(t_s)$ and $-\overline{KS}^{\nu}_{m,n}(t_{m+n-s})$ are equally distributed and asymptotically independent we obtain the assertion.
(ii) Let $n\equiv n(m)\ge m$, $m\in\mathbb{N}$, so that $p=0$. The case $p=1$ can be treated analogously. From (A3.15), Lemma A4.1 and results in [14] we obtain

$$\sup_{t\in(0,1)}\langle\overline{KS}^{\nu}_{m,n}(t)\rangle = \sup_{t\in(0,1)} D^{\nu}_{m,n}(t)\langle -F^{\#,\nu}_{m}(t)\rangle + o_{\mathbb{P}_0^*}(1).$$

With results in [16] we get that $\sup_{t\in(a_n,1-a_n)}|D^{\nu}_{m,n}(t)|\to 1$ in probability for $a_n\in(0,0.5)$ with $na_n\to\infty$ as $n\to\infty$. Therefore,

$$\sup_{t\in(a_n,1-a_n)}\langle\overline{KS}^{\nu}_{m,n}(t)\rangle = \sup_{t\in(a_n,1-a_n)}\langle -F^{\#,\nu}_{m}(t)\rangle + o_{\mathbb{P}_0^*}(1).$$

Now choose $a_n\equiv a_{n(m)}$ with $\lim_{m\to\infty} ma_n = 0$. Conditionally on $\{a_n < X_{1:m}, X_{m:m} < 1-a_n\}$, we get $\sup_{t\in(0,a_n)}|F^{\#,\nu}_{m}(t)| = \sup_{t\in(1-a_n,1)}|F^{\#,\nu}_{m}(t)| = o(1)$. Since $\lim_{m\to\infty}\mathbb{P}_0^*(a_n<X_{1:m}, X_{m:m}<1-a_n) = 1$, we get

$$\sup_{t\in(0,a_n)}|\overline{KS}^{\nu}_{m,n}(t)| = \sup_{t\in(1-a_n,1)}|\overline{KS}^{\nu}_{m,n}(t)| = o_{\mathbb{P}_0^*}(1).$$

Hence,

$$\sup_{t\in(0,1)}\langle\overline{KS}^{\nu}_{m,n}(t)\rangle=\sup_{t\in(0,1)}\langle -F^{**,\nu}_m(t)\rangle+o_{\mathbb{P}_0^*}(1).$$

Altogether, we obtain that the asymptotics of $\sup_{t\in(0,1)}\langle\overline{KS}^{\nu}_{m,n}(t)\rangle$ coincides with $\sup_{t\in(0,1)}\langle -F^{\#,\nu}_m(t)\rangle$ for $p=0$ and $\sup_{t\in(0,1)}\langle G^{\#,\nu}_m(t)\rangle$ for $p=1$, cf. [14] for the one-sample asymptotics. □

Proof of Theorem 4.1. W.l.o.g. let $n\equiv n(m)\geq m$, $m\in\mathbb{N}$. We show that one-sided minP tests based on $\alpha^*_{m,n}$ are asymptotic level α tests, i.e.,

$$\lim_{m\to\infty}\mathbb{P}_0(\min_{s\in I_{m,n}}p_s\leq\alpha^*_{m,n})=\alpha. \tag{A3.17}$$

The two-sided case can be treated in the same way by utilizing the symmetry properties discussed in Section 2.3 in [7].

For $\alpha^{loc}_{m,n}=\alpha^*_{m,n}$ let c^*_s, $s\in I_{m,n}$, be defined by (4.1). Setting $s_0=\lfloor(m+n)\log(m)/m\rfloor$, we split $I_{m,n}$ into $J_1=\{1,\ldots,s_0\}\cup\{m+n-s_0,\ldots,m+n-1\}$ and $J_2=I_{m,n}\setminus J_1$. We get $\mathbb{P}_0(\min_{s\in I_{m,n}}p_s\leq\alpha^*_{m,n})\geq\mathbb{P}_0(\cup_{s\in J_2}\{V_{m,s}<c^*_s\})$ and

$$\mathbb{P}_0(\min_{s\in I_{m,n}}p_s\leq\alpha^*_{m,n})\leq \mathbb{P}_0(\cup_{s\in J_1}\{V_{m,s}<c^*_s\})+\mathbb{P}_0(\cup_{s\in J_2}\{V_{m,s}<c^*_s\}). \tag{A3.18}$$

Now we show

$$\lim_{m\to\infty}\mathbb{P}_0(\cup_{s\in J_1}\{V_{m,s}<c^*_s\})=0. \tag{A3.19}$$

Let $b_m(x^+_\alpha)$ be defined as in Section 3.2, cf. the notation before Theorem 3.2. From Remark 4 in [8] we get $\alpha^*_{m,n}=\Phi(-b_m(x^+_\alpha))(1+o(1))$. Since the distribution of $(V_{m,s_0}-\mathbb{E}[V_{m,s_0}])/\sqrt{\mathbb{V}ar[V_{m,s_0}]}$ converges weakly to a standard normal distribution if $\mathbb{V}ar[V_{m,s_0}]\to\infty$ for $m,n\to\infty$, cf. e.g. [13], we get $c^*_{s_0}=\mathbb{E}[V_{m,s_0}]-b_m(x^+_\alpha)\mathbb{V}ar[V_{m,s_0}](1+o(1))$, which immediately implies $c^*_{s_0}\leq\log(m)$ for sufficiently large m-values. Let s^*_i, $i=1,\ldots,c^*_{s_0}$, be the indexes with $c^*_s=i$ for $s\in[s^*_i,s^*_{i+1})$. For any $s\leq s_0$ there exists an $i\in\{1,\ldots,c^*_{s_0}\}$ such that $s\in[s^*_i,s^*_{i+1})$. Moreover, $\{V_{m,s}<c^*_s\}\subseteq\{V_{m,s^*_i}<c^*_{s^*_i}\}$. Altogether, we get

$$\mathbb{P}_0(\cup_{s=1}^{s_0}\{V_{m,s}<c^*_s\})=\mathbb{P}_0(\cup_{i=1}^{c^*_{s_0}}\{V_{m,s^*_i}<c^*_{s^*_i}\})\leq\log(m)\alpha^*_{m,n}\to0$$

as $m\to\infty$. Analogously, we get $\lim_{m\to\infty}\mathbb{P}_0(\cup_{s=m+n-s_0}^{m+n-1}\{V_{m,s}<c^*_s\})=0$ and hence (A3.19). Together with (A3.18) we obtain

$$\lim_{m\to\infty}\mathbb{P}_0(\min_{s\in I_{m,n}}p_s\leq\alpha^*_{m,n})=\lim_{m\to\infty}\mathbb{P}_0(\cup_{s\in J_2}\{V_{m,s}<c^*_s\}).$$

With the same arguments as in the proof of Theorem 1 in [8] for the one-sample case, we get (A3.17) and the remaining assertions. □

A4. Technical results. The following two lemmas characterize the distribution of the auxiliary process $D^{\nu}_{m,n}(t)$, $t \in [t_1, t_{m+n})$, defined in (A3.4).

LEMMA A4.1. *Let $\mathbb{P}^*_0$ be defined in (A3.1). For $D^{\nu}_{m,n}$ defined in (A3.4) we get*

$$\sup_{t\in[t_1,t_{m+n})} D^{\nu}_{m,n}(t) = O_{\mathbb{P}^*_0}(1).$$

PROOF. It holds

$$\mathbb{P}^*_0\Big(\sup_{t\in[t_1,1]} t/\hat{H}_{m+n}(t) \geq \lambda\Big) \leq \lambda \exp(1-\lambda), \tag{A4.1}$$

cf. [1] or p.137 in [15]. Noting that $\sup_{t\in[t_1,1]} t/\hat{H}_{m+n}(t)$ has the same distribution as $\sup_{t\in[0,t_{m+n})}(1-t)/(1-\hat{H}_{m+n}(t))$, we obtain

$$\mathbb{P}^*_0\Big(\sup_{t\in[0,t_{m+n})} (1-t)/(1-\hat{H}_{m+n}(t)) \geq \lambda\Big) \leq \lambda \exp(1-\lambda). \tag{A4.2}$$

Moreover, since $t/\hat{H}_{m+n}(t) \geq 1$ if and only if $(1-t)/(1-\hat{H}_{m+n}(t)) \leq 1$, we get for any $\lambda > 1$ that

$$\{D_{m,n}(t) \geq \lambda\} \subseteq \{t/\hat{H}_{m+n}(t) \geq \lambda\} \cup \{(1-t)/(1-\hat{H}_{m+n}(t)) \geq \lambda\}.$$

Together with (A4.1) and (A4.2) we get

$$\mathbb{P}^*_0\Big(\sup_{t\in[t_1,t_{m+n})} D^{\nu}_{m,n}(t) \geq \lambda\Big) \leq 2\lambda^{1/\nu} \exp(1-\lambda^{1/\nu}), \tag{A4.3}$$

hence the assertion follows. □

LEMMA A4.2. *Let $\nu > 0$. For $D^{\nu}_{m,n}$, $T_{m,n}$ and $\mathbb{P}^*_0$ defined in (A3.4), in (A3.8) and (A3.1), respectively, we get*

$$\sup_{t\in T_{m,n}\cap[t_1,t_{m+n})} D^{\nu}_{m,n}(t) = 1 + O(\sqrt{\log_2(m+n)/\log(m+n)}) \;\; \mathbb{P}^*_0 - a.s..$$

The proof follows with results in [4], also cf. the proof of Lemma 5 in [12].

The next theorem shows that weighted empirical processes based on one sample can be approximated by suitable sequences of weighted Brownian bridges.

THEOREM A4.1. *Let* $G_n^{*,\nu}(t)$ *and* $F_m^{*,\nu}(t)$ *be defined in* (A3.3) *and let* $\mathbb{P}_0^*$ *be defined in* (A3.1). *Then there exists a probability space* $(\Omega, \mathcal{A}, \mathbb{P}_0^*)$ *with two mutually independent sequences of weighted Brownian bridges* $(\mathbb{B}_n^{1,\nu}(t) : t \in (0,1))$, $n \in \mathbb{N}$, *and* $(\mathbb{B}_m^{2,\nu}(t) : t \in (0,1))$, $m \in \mathbb{N}$, *such that*
(*a*) *for* $\nu \in [0, 0.5)$

$$\mathbb{P}_0^*\Big(\sup_{t\in(0,1)} |G_n^{*,\nu}(t) - \mathbb{B}_n^{1,\nu}(t)|\Big) = o_{\mathbb{P}_0^*}(1), \ \ \mathbb{P}_0^*\Big(\sup_{t\in(0,1)} |F_m^{*,\nu}(t) - \mathbb{B}_m^{2,\nu}(t)|\Big) = o_{\mathbb{P}_0^*}(1),$$

and,
(*b*) *for* $\nu = 0.5$

$$\mathbb{P}_0^*\Big(\sup_{t\in T_n} |G_n^{*,0.5}(t) - \mathbb{B}_n^{1,0.5}(t)|\Big) = o(1/\sqrt{\log_2(n)}) \ \ \mathbb{P}_0^* - a.s.,$$

$$\mathbb{P}_0^*\Big(\sup_{t\in T_m} |F_m^{*,0.5}(t) - \mathbb{B}_m^{2,0.5}(t)|\Big) = o(1/\sqrt{\log_2(m)}) \ \ \mathbb{P}_0^* - a.s.,$$

where T_m *is defined before Theorem 3.2.*

PROOF. Part (*a*) follows by applying Theorem 4.2.1 in [2], also cf. Theorem 12.20 in [5]. Part (*b*) follows by (*c*) on p. 601 in [15]. □

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